The geometric structure of 0+1d-QCD and its use in Monte-Carlo simulations

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Lattice QCD on Lefschetz Thimbles, Trento 2017

Taking 0+1d-QCD to the Lattice

The euclidean Action of QCD reads naturally

$$\begin{aligned} S^E &= \int \mathrm{d}^4x \frac{i}{4} F_{\mu\nu}F_{\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi}^f (\gamma_\mu^E(\partial_\mu + ig_0 A_\mu) + m_f)\psi^f, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig_0[A_\mu, A_\nu]. \end{aligned}$$

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In one dimension, the Gauge-Part naturally vanishes and we are left with the fermionic part

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Using staggered fermions, the discretized action for $a = 1$ is

$$\hat{S}_F = \frac{1}{2} \sum_{n=0}^{N_\tau - 1} \sum_{f=1}^{N_f} \bar{\chi}^f(n) \Big(U(n) \chi^f(n+1) - U^\dagger(n-1) \chi^f(n-1) + 2 m_f \chi^f(n) \Big)$$

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... and adding chemical potential μ according to (Karsch and Hasenfratz), we have finally

$$\hat{S}_F(\mu) = \frac{1}{2} \sum_{n=0}^{N_\tau - 1} \sum_{f=1}^{N_f} \bar{\chi}^f(n) \left(e^{\mu_f} U(n) \chi^f(n+1) - e^{-\mu_f} U^\dagger(n-1) \chi^f(n-1) + 2 m_f \chi^f(n) \right).$$

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Integrating out the fermion fields in the partition sum, we have

$$Z(N_\tau, \mu) = \int \mathrm{d}U \mathrm{d}\bar{\chi} \mathrm{d}\chi e^{-\hat{S}_F(N_\tau, \mu)} = \int \mathrm{d}U \prod_{f=1}^{N_f} \det M_f[U]$$

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with the Fermion Matrix

$$\begin{pmatrix} m_f\mathbb{I} & \frac{1}{2}e^{\mu_f}U(0) & 0 & \dots & 0 & \frac{1}{2}e^{-\mu_f}U^\dagger(N_\tau-1) \\ -\frac{1}{2}e^{-\mu_f}U^\dagger(0) & m_f\mathbb{I} & \frac{1}{2}e^{\mu_f}U(1) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & \\ -\frac{1}{2}e^{\mu_f}U(N_\tau-1) & 0 & \dots & -\frac{1}{2}e^{-\mu_f}U^\dagger(N_\tau-2) & m_f\mathbb{I} & \frac{1}{2}e^{\mu_f}U(N_\tau-2) \end{pmatrix}$$

The Reduced Fermion Matrix

Using a Matrix identity fully proved last year by (A. Ammon et al.), $\det M_f$ is reduced to a determinant of a 3x3 Matrix:

$$\begin{aligned}\det(M_f[U]) &= \frac{1}{2^{3N_\tau}} \det\left(2\cosh(N_\tau \sinh^{-1}(m_f))\mathbb{I} + e^{N_\tau \mu_f} P + e^{-N_\tau \mu_f} P^\dagger\right) \\ P &= \prod_{n=0}^{N_\tau - 1} U(n).\end{aligned}$$

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From now on

- $N_f = 1$,
- $\mu_c := \sinh^{-1}(m)$,
- $A := 2\cosh(N_\tau \mu_c)$ and leave out $\frac{1}{2^{3N_\tau}}$.

$$\Rightarrow \quad M = A\mathbb{I} + e^{N_\tau \mu} P + e^{-N_\tau \mu} P^\dagger$$

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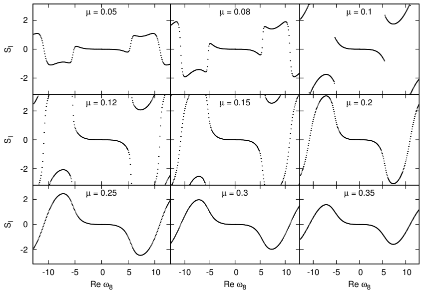

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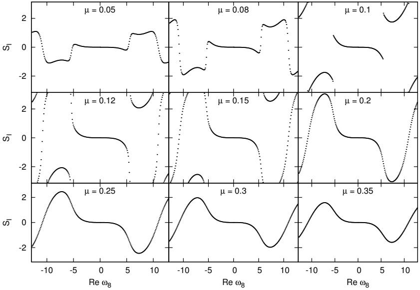

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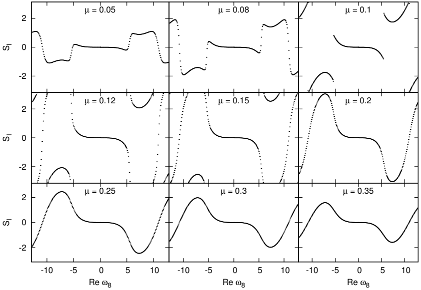

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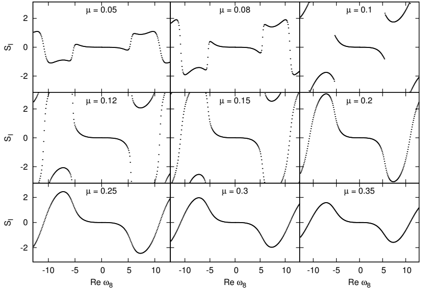

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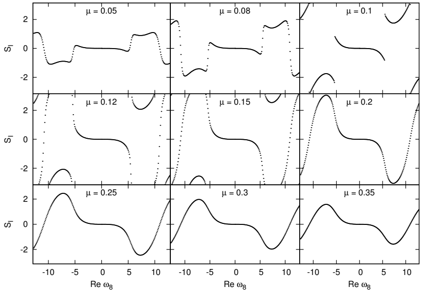

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The most interesting region is for small μ_c and $0 < \mu \lesssim 3\mu_c$.

Complexification

- We use the representation of the SU(3)-matrices P as exponentials of Gell-Mann-Matrices

$$P = \exp\left[\sum_{k=1}^{8} \omega_k T^k\right] \quad T^k = -\frac{i}{2}\lambda^k \; k = 1 \ldots 8$$

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3. Especially S is only *locally holomorphic*!

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- This equation is non-trivial, since $\frac{\partial S}{\partial \omega_k}$ is not invariant under similarity transformations of P...
- But we can argue for some obvious solutions!

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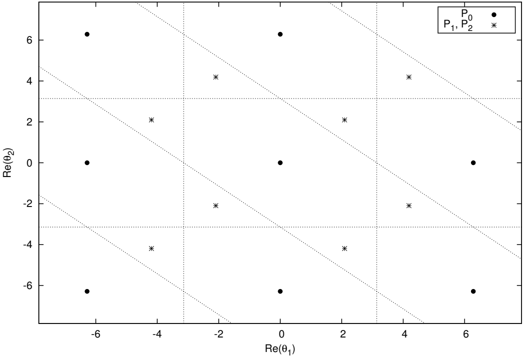
Including the divergent regions, we have the following picture:



The other critical points

For P being diagonal, we can write two equations, which can be analyzed by algebra:

$$\frac{e^{N_\tau \mu} x - e^{-N_\tau \mu} x^{-1}}{2\cosh(N_\tau \mu_c) + e^{N_\tau \mu} x + e^{-N_\tau \mu} x^{-1}} = \frac{e^{N_\tau \mu} y - e^{-N_\tau \mu} y^{-1}}{2\cosh(N_\tau \mu_c) + e^{N_\tau \mu} y + e^{-N_\tau \mu} y^{-1}} \tag{1}$$

$$= \frac{e^{N_\tau \mu} x^{-1} y^{-1} - e^{-N_\tau \mu} xy}{2\cosh(N_\tau \mu_c) + e^{N_\tau \mu} x^{-1} y^{-1} + e^{-N_\tau \mu} xy}, \tag{2}$$

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with $x = e^{i\theta_1}$ and $y = e^{i\theta_2}$.
The fractions can be factorized into (e.g. for x)

$$\frac{(x + e^{-N_\tau\mu})(x - e^{-N_\tau\mu})}{(x + e^{-N_\tau(\mu-\mu_c)})(x + e^{-N_\tau(\mu+\mu_c)})},$$

which exhibits some specialty in the chiral limit ($\mu_c = 0$). There, they reduce to

$$\frac{x - e^{-N_\tau\mu}}{x + e^{-N_\tau\mu}}$$

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- But being points, where $S_R \to \infty$ before they have a very large action and seem not to contribute numerically.
- Besides, they're clearly not on $\mathrm{SU}(3)$ like the others and the intersection number needs further investigation.

The other critical points

By Multipying the denominators, one just gets for (1) and (2)

$$x - y = 0 \text{ and } y^2 x - 1 = 0 \Rightarrow x^3 - 1 = 0,$$

So, in the chiral limit, the center elements are the only critical points! But for $\mu_c > 0$, one gets additional solutions that pop up, because the domains, where S is singular, split apart with rising μ_c.

- But being points, where $S_R \to \infty$ before they have a very large action and seem not to contribute numerically.
- Besides, they're clearly not on $\mathrm{SU}(3)$ like the others and the intersection number needs further investigation.
- The Hessian does not have the typical structure, which is linked to S being non-holomorphic everywhere.

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... with $H^{kl} = h_\gamma\delta^{kl}$, we have as solutions

$$\lambda = |h_\gamma|, \quad \rho_\lambda^k = ce^k \;\; \text{with} \;\; c = \sqrt{\frac{h_\gamma^*}{|h_\gamma|}}.$$

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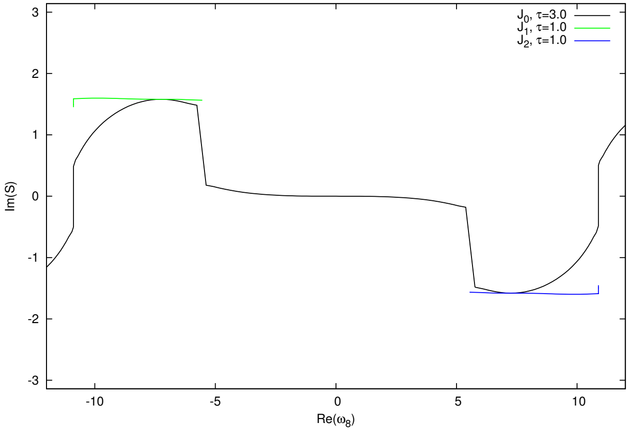
For $\gamma = \pm\frac{2\pi}{3}$, this is not the case.



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The decomposition of the partition sum is

$$Z = \int_{\mathrm{SU}(3)} \mathrm{d}P e^{-S[P]} = \sum_{\sigma=0}^{2} \int_{\mathcal{J}_k} \mathrm{d}P e^{-S[P]} := \sum_{\sigma=0}^{2} Z_\sigma$$

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$$\Rightarrow \quad Z \approx \sum_{\sigma=0}^{2} \int \prod_{k=1}^{8} \mathrm{d}\omega_k e^{-S[P_\sigma] - \frac{1}{2} \sum_k \left. \frac{\partial^2 S}{\partial \omega_k \partial \omega_k} \right|_{P_\sigma} \omega_k^2}$$

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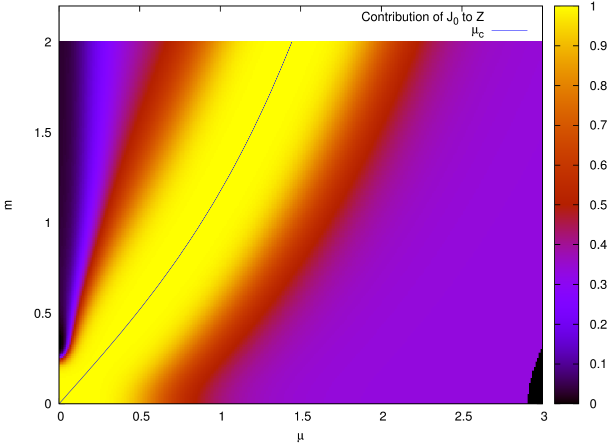

$$\frac{|Z_0|}{\sum_\sigma |Z_\sigma|}$$

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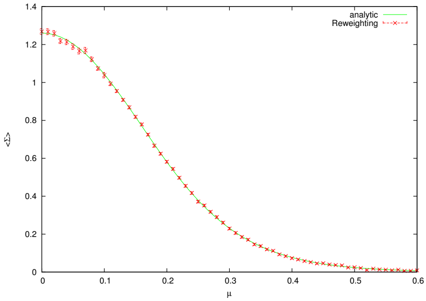

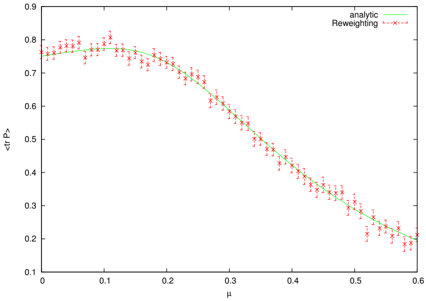

This is done with 20000 Updates at $N_\tau = 4$ and $m = 0.1$.

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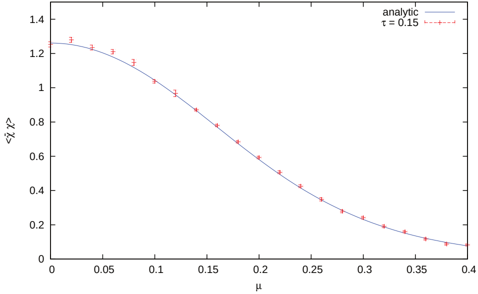

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