$\pi - \pi$ Scattering from Lattice QCD with Isospin $I = 2$, $I = 1$ and $I = 0$

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Projects A2 and B5 of Sino-German CRC 110

ECT* Workshop, February 2016

Particle Scattering in Lattice QCD

- most of the hadrons are resonances

- non-perturbative determination of resonance and scattering properties from first principles highly valuable

$\Rightarrow$ Lattice QCD

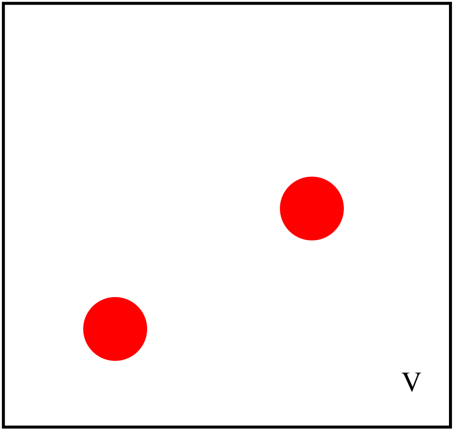
$\Rightarrow$ need to go beyond single hadron masses and matrix elements of QCD stable states

- unfortunately: direct determinations in Euclidean time very difficult (impossible)

  [Maiani and Testa (1990)]

$\Leftarrow$ how to treat such systems with Lattice QCD?

... make use of finite size effects!



- for $V \to \infty$:

$\Rightarrow$ interaction probability very low

$\Rightarrow$ $E_{2p}(p = 0) = 2M_{1p}$

- for finite V:

$\Rightarrow$ interaction probability rises

$\Rightarrow$ $E_{2p}(p = 0)$ receives corrections $\propto 1/V$

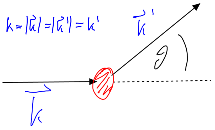
- Lüscher: correction in $1/V$ related to scattering properties!

[Lüscher, 1986]

- assume a finite range potential $V(r) = 0 \quad \forall \quad r > R$

- scattering amplitude in the partial wave expansion

$$f_k(\theta) = -\frac{8\pi}{M}\sum_{\ell=0}^{\infty}(2\ell+1)\ f_\ell(k)\ P_\ell(\cos\theta)$$



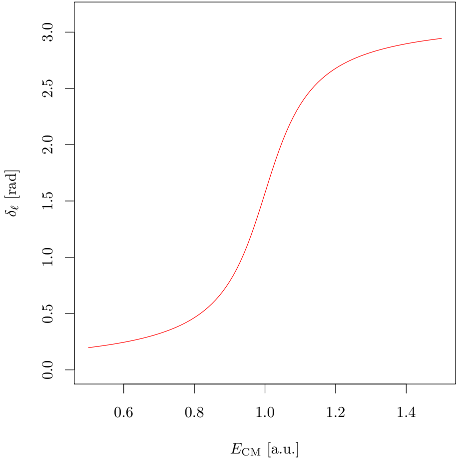
with partial wave amplitude and phase shift δ_ℓ

$$f_\ell(k) = \frac{1}{k\cot\delta_\ell(k) - ik}$$

- Lüscher: $\delta_\ell(k)$ can be extracted from finite volume effects

Goal

- would like to map out phase shift
- then extract mass and width (or pole position)
- however: Lüscher method will give only discrete points
- need various volumes
- can also use different reference frames



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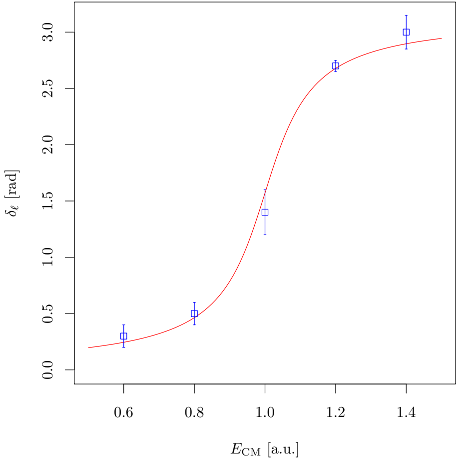


- expectation value of operator $\mathcal{O}$

$$\langle\mathcal{O}\rangle = \int \mathcal{D}A\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{O}\, \exp(\, iS[\bar{\psi},\psi,A]\,)$$

- Ken Wilson noticed: rotate $t \to i\tau$ to Euclidean space-time

$$\langle\mathcal{O}\rangle = \int \mathcal{D}A\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{O}\, \exp(\, -S_\mathrm{E}[\bar{\psi},\psi,A]\,)$$

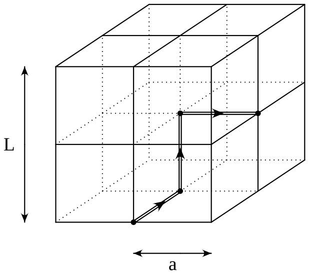
- allows to apply Monte-Carlo methods
  interpreting $\exp(-S_\mathrm{E})$ as a probability density

- can still obtain Minkowski space quantities

  [Osterwalder and Schrader (1973,1975)]

$\rightarrow$ source of statistical errors

- lattice regularisation: discretise space-time
  - hyper-cubic $L^3 \times T$-lattice with lattice spacing a
  - $\Rightarrow$ momentum cut-off: $k_{\max} \propto 1/a$
  - derivatives $\Rightarrow$ finite differences
  - integrals $\Rightarrow$ sums
  - gauge potentials A_μ in $G_{\mu\nu}$ $\Rightarrow$ link matrices U_μ ('●⟹●')



sources of systematic errors

- need to remove the cut-off, continuum limit $a \rightarrow 0$
- quark mass values $m_u = m_d = m_\ell \propto M_\pi^2$ in general not physical guidance by chiral perturbation theory

[Weinberg (1979), Gasser and Leutwyler (1984,1985)]

- $2+1+1$ quark flavour ensembles from ETM Collaboration
  $m_u = m_d < m_s < m_c$ Wilson twisted mass fermions

  [Frezzotti, Rossi, (2004); ETMC, R. Baron et. al., JHEP 06 111 (2010)]

- improved scaling: $\propto \mathcal{O}(a^2)$
  flavour symmetry broken at finite lattice spacing values

- charged pion masses range from $\approx 230\ \mathrm{MeV}$ to $\approx 500\ \mathrm{MeV}$

- $L \geq 3\ \mathrm{fm}$ and $M_\pi \cdot L \geq 3.5$ for most ensembles

- bare m_s and m_c fixed for each lattice spacing

- three lattice spacings (A, B and D ensembles):
  $a_A = 0.086\ \mathrm{fm}$, $a_B = 0.078\ \mathrm{fm}$ and $a_D = 0.061\ \mathrm{fm}$

- special smearing method: stochastic Laplacian Heaviside

  [Peardon et al, (2009), Morningstar et al, (2011)]

Ensemble-Details

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Warm-up: $\pi - \pi$ Scattering with $I = 2$

- at small momenta $k \to 0$ use effective range expansion

$$k^{2\ell+1} \cot \delta_\ell = \frac{1}{a_\ell} - \frac{r_\ell k^2}{2} + \ldots$$

scattering length a_ℓ and effective range r_ℓ

$\Rightarrow$ only S-waves ($\ell = 0$) contribute (to a good approximation)

- Lüscher:

$$\delta E \;\equiv\; E_{2p}(L) - 2M_{1p} \;=\; -\frac{4\pi a_0}{M_{1p} L^3} \left\{ 1 + \mathcal{O}\left(\frac{1}{L}\right) \right\}$$

- translates directly to Quantum Field-theory for $\pi\pi$ systems with $I = 0, 2$ and πN with $I = 1/2, 3/2$

- let $O(x,t)$ be an operator with quantum numbers of a given state
- for instance for the pion it is given by

$$O_\pi(x,t) \;=\; \bar{u} i\gamma_5 d(x,t)\,, \qquad O(t) \;=\; \sum_x O_\pi(x,t)$$

  projected to zero momentum

- create pion at time 0 and annihilate at t

$$C_\pi(t) \;=\; \langle O_\pi(0)\; O_\pi^\dagger(t)\rangle \;\propto\; \sum_n \langle 0|O(0)|n\rangle\langle n|e^{-Ht}O(0)e^{Ht}|0\rangle$$

$$\propto \sum_n |\langle 0|\; O\; |n\rangle|^2\; e^{-(E_n-E_0)t}$$

- for large Euclidean times $t\to\infty$

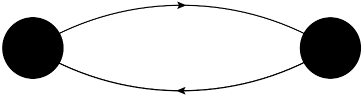
$$\lim_{t\to\infty} C_\pi(t) \;\propto\; |\langle 0|\; O\; |1\rangle|^2 e^{-(E_1-E_0)t} = |\langle 0|\; O\; |\pi\rangle|^2 e^{-M_\pi t}$$

Energy Levels on the Lattice

- correlation functions are constructed from contractions

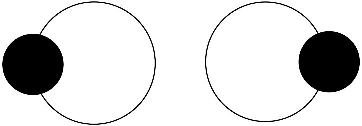
- correlation function

$$C(t) = \langle O(0)\ O^\dagger(t)\rangle$$



- in general:
  - quark connected
  - quark disconnected

  contribution



- disconnected: usually very noisy

- connected much better

- similar for two-particle operator:

$$O_{\pi^+\pi^+}(x,t) \;=\; \bar{u}i\gamma_5 d\; \bar{u}i\gamma_5 d(x,t)$$

and

$$C_{\pi\pi}(t) \;=\; \langle O_{\pi^+\pi^+}(0)\; O^\dagger_{\pi^+\pi^+}(t)\rangle$$

- no quark disconnected contribution
- extract $E_{\pi\pi}$ for $t \;\to\; \infty$ from $C_{\pi\pi} \propto \exp(-E_{\pi\pi}t)$
- compute scattering length a_0 from

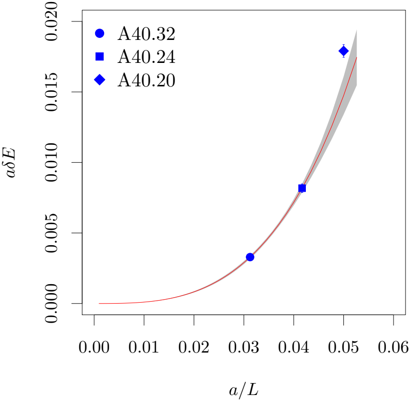
$$\delta E = E_{\pi\pi} - 2M_\pi \;\propto\; -\frac{4\pi a_0}{M_\pi L^3}(1 + ...)$$

- some technical details have been skipped, like e.g. thermal state pollutions, which need to be treated appropriately

Lüscher formula (known constants c_i)

$$\delta E = -\frac{4\pi a_0}{M_\pi L^3}\left(1 + c_1 \frac{a_0}{L} + c_2 \frac{a_0^2}{L^2}\right) + \mathcal{O}(L^{-6})\,,$$

- valid, if other FS corrections small
- three ensembles with identical parameters but L
- smallest L deviates a few sigma
- A40.20 probably too small L
- all other ensembles have comparably larger L-values



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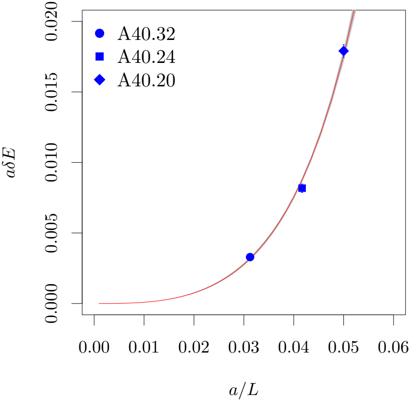
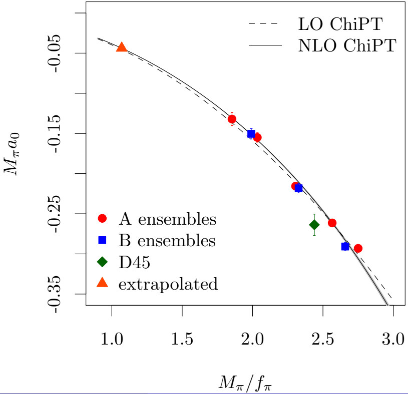


- ChiPT formula at NLO [Beane et al, (2005,2007)]

$$M_\pi a_0 = -\frac{M_\pi^2}{8\pi f_\pi^2}\left\{1 + \frac{M_\pi^2}{16\pi^2 f_\pi^2}\left[3\ln\frac{M_\pi^2}{f_\pi^2} - 1 - \ell_{\pi\pi}^{I=2}(\mu_R = f_{\pi,\mathrm{phys}})\right]\right\}$$

- functional form highly constraining
- surprisingly small deviations from LO ChiPT
- lattice artifacts small (in fact $\mathcal{O}(a^2 m_q)$)
- see JHEP 1509 (2015) 109



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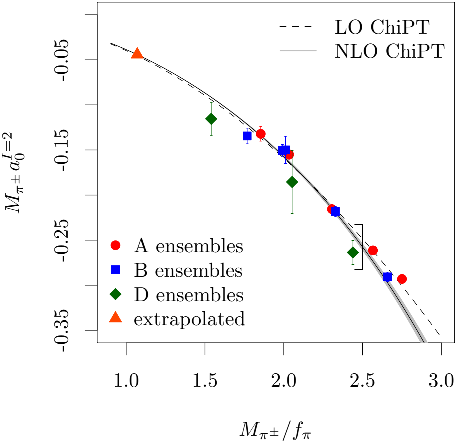
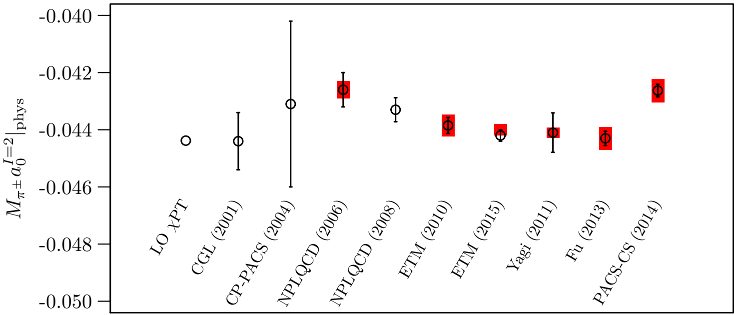
- functional form highly constraining
- surprisingly small deviations from LO ChiPT
- lattice artifacts small (in fact $\mathcal{O}(a^2 m_q)$)
- additional data for finest lattice spacing



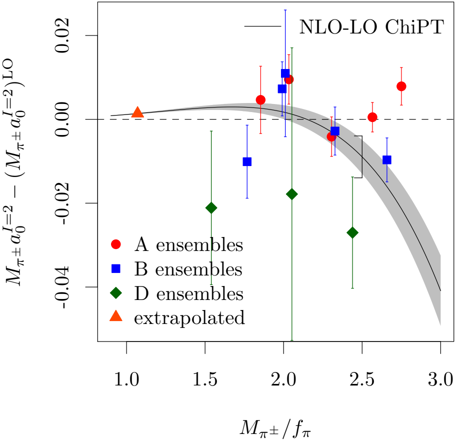


- result:

$$M_\pi a_0^{I=2} = -0.0442(2)_{\rm stat}(^{+4}_{-0})_{\rm sys}\,, \qquad \ell_{\pi\pi}^{I=2} = 3.79(0.61)_{\rm stat}(^{+1.34}_{-0.11})_{\rm sys}$$

[ETMC, Helmes, CU, et al, (2025)]

- LO ChiPT (parameter-free) subtracted
- any single point not significantly different from 0
- explains the large uncertainty on $\ell_{\pi\pi}^{I=2}$
- significantly higher precision needed to sort out



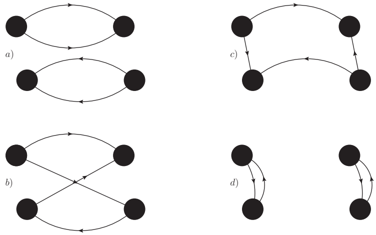
Challenge: $\pi - \pi$ Scattering with $I = 0$

- two-particle operator for $I = 0$ (with $\Gamma = \gamma_5, 1$):

$$\mathcal{O}_\Gamma^{I=0} = \frac{1}{\sqrt{3}}(\mathcal{O}_\Gamma^+\mathcal{O}_\Gamma^- + \mathcal{O}_\Gamma^-\mathcal{O}_\Gamma^+ + \mathcal{O}_\Gamma^0\mathcal{O}_\Gamma^0)\,, \quad \text{e.g.} \quad \mathcal{O}_{\gamma_5}^{\pm,0} \equiv \pi^{\pm,0}$$

- isospin limit: only four diagrams contribute

$$C_{\pi\pi}(t-t') = \langle \mathcal{O}_{\gamma_5}^{I=0}(t')(\mathcal{O}_{\gamma_5}^{I=0})^\dagger(t)\rangle$$
$$= D(t) + \frac{1}{2}X(t) - 3B(t) + \frac{3}{2}V(t)$$



- in the elastic region sufficient to determine δE from $C_{\pi\pi}$!
- $a_0^{I=0}$ in principle from same analysis as for $I = 2$

$I = 0$ in Twisted Mass Lattice QCD

- Twisted Mass Lattice QCD explicitly breaks isospin symmetry at finite lattice spacing values

$\Rightarrow$ cannot project to states with $I = 0$

see also [Buchoff et al., (2009)]

- way out $\Rightarrow$ valence action with explicit isospin symmetry

- con: have to deal with lattice artefacts from unitarity violations
here: mixing with lower lying states (due to vacuum diagram)

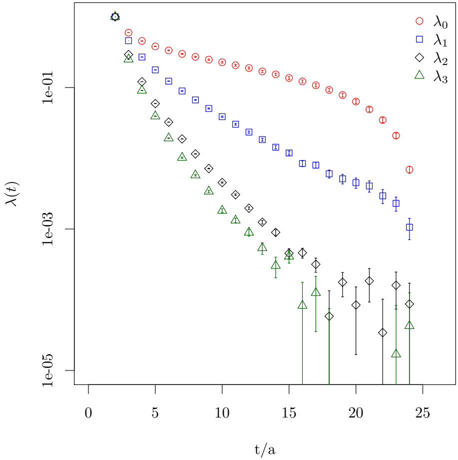
$\Rightarrow$ use larger operator basis $\mathcal{O}^{\mathrm{I}=0}_{i\gamma_5}, \mathcal{O}_\sigma, \mathcal{O}^{\mathrm{I}=0}_{1}, \mathcal{O}_{\pi^0_u}$

- apply generalised eigenvalue problem (GEVP) to identify state of interest

[Michael and Teasdale (1983); Lüscher and Wolff (1990)]

- ensemble with $L = 24$ and high statistics
- four states clearly identifiable
- the higher the state, the worse the signal
- we need the third state ($\lambda_2(t)$)

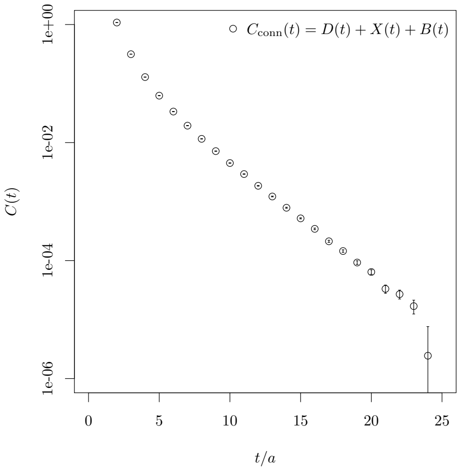
$\Rightarrow$ problematic on ensembles with less statistics



- can we avoid this complication of mixing with lower lying states?

- complication comes solely from vacuum diagram $V(t)$
- contribution of D and V suppressed in ChiPT
  [Guo, Liu, Meißner (2013)]
- was found to be negligible in only ever lattice computation
  [Fu, (2013)]
- we can test this on our high statistics ensemble

$\Rightarrow$ within errors no contribution!



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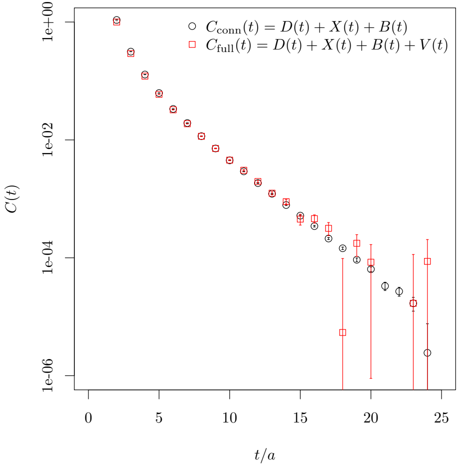
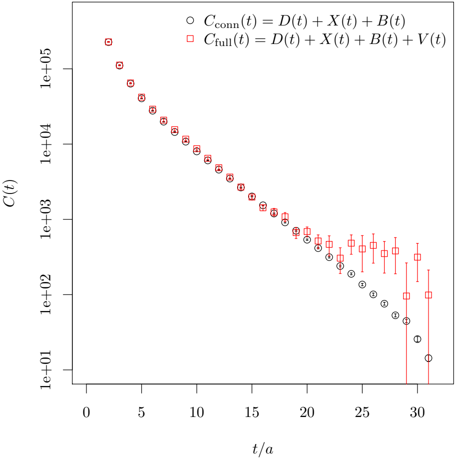


- Cross-check on PACS-CS ensemble
  $M_\pi \approx 300\ \mathrm{MeV}$
  [PACS-CS: Aoki et al, (2009)]
- $N_f = 2 + 1$ clover fermions
  u/d isospin symmetry explicit
- separate connected and disconnected contributions
- disconnecteds have negligible contribution
  until signal is lost in noise



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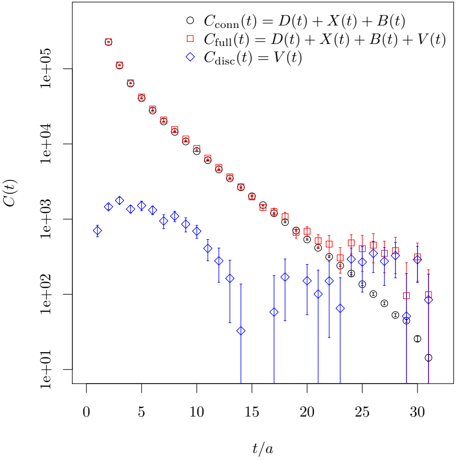


- in ChiPT at NLO $M_\pi a_0^{I=0}$ depends on M_π/f_π

[Bijnens et al, (1997)]

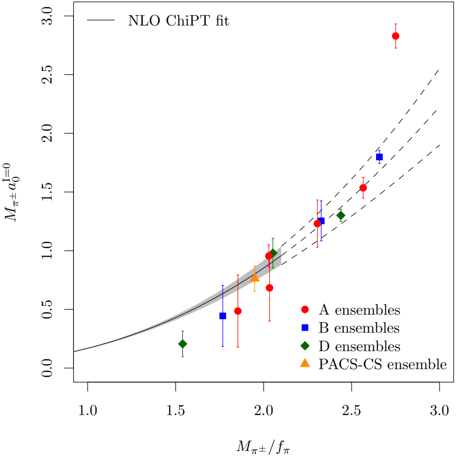
$$M_\pi a_0^{I=0} \;=\; \frac{7M_\pi^2}{16\pi f_\pi^2}\left[1 - \frac{M_\pi^2}{16\pi^2 f_\pi^2}\left(9\ln\frac{M_\pi^2}{f_\pi^2} - 5 - \ell_{\pi\pi}^{\mathrm{I=0}}(\mu = f_{\pi,\mathrm{phys}})\right)\right]$$

- at LO parameter free, at NLO only one parameter

$$\ell_{\pi\pi}^{\mathrm{I=0}}(\mu) \;=\; \frac{40}{21}\bar{\ell}_1 + \frac{80}{21}\bar{\ell}_2 - \frac{5}{7}\bar{\ell}_3 + 4\bar{\ell}_4 + 9\ln\frac{M_\pi^2}{\mu^2}$$

- can be fit to the data, highly constraining

- excellent agreement with result on PACS-CS ensemble (not included in the fit)
- when does the σ become stable?
  [Hanhart et al. (2008,2014), Pelaez and Rios (2010)]
- systematic error from full analysis
- no lattice artifacts within errors
- preliminary result



$$M_\pi a_0^{I=0} = 0.197(5)_{\mathrm{stat}}(+23)_{\mathrm{sys}}\,, \qquad \ell_{\pi\pi}^{\mathrm{I}=0} = 28(5)_{\mathrm{stat}}(+20)_{\mathrm{sys}}$$

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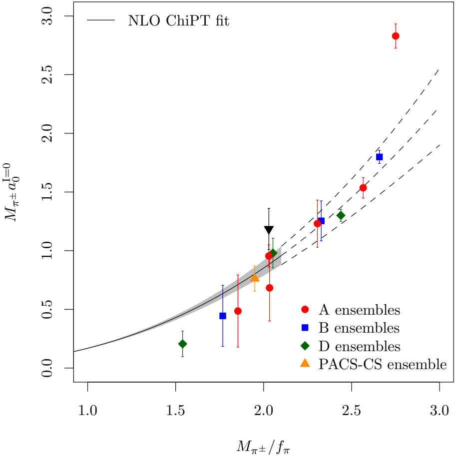

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Preliminary Summary $\pi - \pi$ for $I = 0$

	$M_\pi a_0^{\mathrm{I}=0}$	$\ell_{\pi\pi}^{\mathrm{I}=0}$
Lattice:		
This work	0.197(5)(+23)	28(5)(+20)
Fu (2013)	0.214(4)(7)	43.2(3.5)(5.6)
ChiPT:		
CGL (2001)	0.220(5)	48.5(4.3)
Weinberg (1966)	0.1595(5)	–
Experiment:		
NA48/2 (2011)	0.221(5)(2)	49.3(4.1)(1.3)
E865 (2003)	0.216(13)(2)	45.0(11.2)(3.5)

Outlook: $\pi - \pi$ Scattering with $I = 1$
The ρ-Resonance

- so far: limit of small scattering momenta $k^2 \to 0$
- now: study phaseshift as a function of energy
- ρ decays to $\pi\pi$ in P wave
  (we assume that higher partial waves can be neglected)

$\Rightarrow$ pions must carry momentum

- make sure $M_\rho > 2M_\pi(\vec{p})$
- at $E = 2M_K$ a second channel opens

The ρ-Resonance al a Lüscher

- in the CMF Lüschers relation then looks as follows [Lüscher, (1986-1991)]

$$\cot \delta_1 = \frac{\mathcal{Z}^{\vec{0}}_{00}(q^2)}{\pi^{3/2} q}$$

- $\mathcal{Z}$ the generalised Lüschers function

- and the scattering momentum

$$q^2 = \left(\frac{E^2_{\mathrm{CM}}}{4} - M^2_\pi\right) \frac{L^2}{(2\pi)^2}$$

- E_{CM} the interacting energy level in the CMF

- we increase the number of points by using different reference frames where similar relations hold

see e.g. [Gockeler et al., (2012)]

- as a start we use the operators

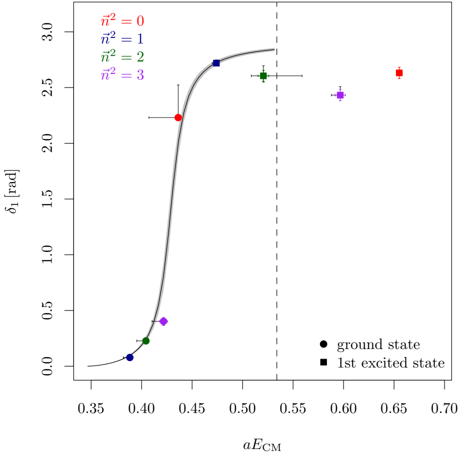
$$\mathcal{C}(\vec{P}, t - t') = \langle\, \vec{O}(t') \,\otimes\, \vec{O}^\dagger(t)\,\rangle\,, \qquad \vec{O}(t) = (\pi\pi(\vec{P}, t), \rho^0(\vec{P}, t))^t\,.$$

- with $\rho^0(\vec{P}, t)$ a single and $\pi\pi(\vec{P}, t)$ a two particle operator
- $\vec{P}$ the total momentum in given reference frame
- we use momenta $\vec{P} = 2\pi\vec{n}/L$ with $\vec{n}^2 = 0, 1, 2, 3, ...$
- solve generalised eigenvalue problem and extract interacting energy levels

Describe phaseshift data using effective range formula [Brown, Gobble, (1968)]

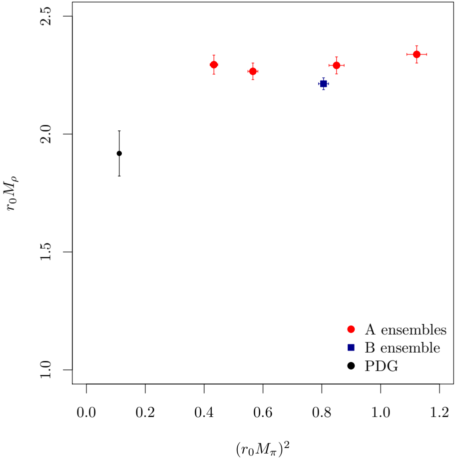
$$\tan \delta_1(E_{\mathrm{CM}}) = \frac{g_{\rho\pi\pi}^2}{6\pi} \frac{\tilde{q}^3}{E_{\mathrm{CM}}(M_\rho^2 - E_{\mathrm{CM}}^2)}$$

- two parameters $g_{\rho\pi\pi}$ and M_ρ
- threshold at $2M_K$
- example for ensemble A60.24
- four reference frames
- dip around $2M_K$ observed on all ensembles



- a few A ensembles and one B ensemble
- r_0 a hadronic scale (the Sommer parameter)
- looks reasonable
- we need a bigger matrix to better control the energy level extraction

$\Rightarrow$ currently ongoing!



Preliminary: ρ-Resonance Mass and $g_{\rho\pi\pi}$ as a Function of M_π^2

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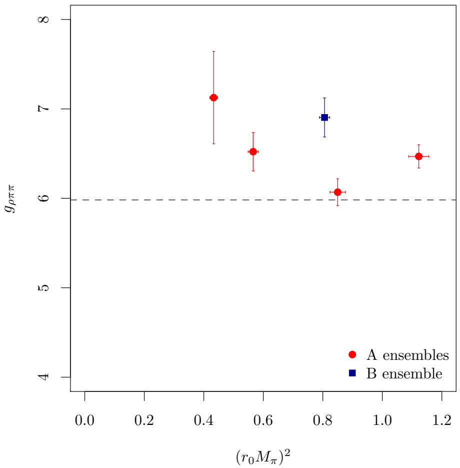

Summary:

- very precise $N_f = 2 + 1 + 1$ lQCD results for $I = 2$ $\pi - \pi$ scattering
- first $I = 0$ $\pi - \pi$ scattering results with multiple lattice spacings
- preliminary results for ρ-resonance mass and coupling

Outlook:

- phaseshifts for $I = 0, 2$ as a function of scattering energy
- fully include disconnected contributions in $\pi - \pi$ $I = 0$
- σ-resonance pole position

Thanks to ...

- the lattice QCD group in Bonn:
  C. Helmes, C. Jost, B. Knippschild, B. Kostrzewa, L. Liu, K. Ottnad, M. Werner
- our friends in Beijing: C. Liu, Z. Wang
- the DFG funding this project in the Sino-German CRC 110
- R. Frezzotti, C. Michael, U.-G. Meißner, A. Rusetsky
- the ETM collaboration
- ... and for your attention!