O(N) spin model with Nienhuis action

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- 2 alternative loop formulations, worm algorithm
- data for 2D O(3) step scaling function, Nienhuis action
- continuum extrapolation

$\mathbf{O}(N)$ model

partition function for standard lattice action:

$$Z = \int \left[\prod_z d\mu(s(z)) \right] \mathrm{e}^{\beta \sum_{l=\langle x\, y\rangle} s(x)\cdot s(y)}, \quad s \in S_{N-1}$$

Nienhuis truncation (same universality class?!):

$$Z = \int \left[\prod_z d\mu(s(z)) \right] \prod_l \left[1 + \tilde{\beta} s(x) \cdot s(y)\right]$$

- exactly solved in $D = 2$ for $-2 \leqslant N \leqslant 2$ on honeycomb lattice
    - in these cases critical region covered for $\tilde{\beta} \leqslant 1$
- includes Kosterlitz Thouless
- Ising, $N = 1$: equivalent for $\tilde{\beta} = \tanh(\beta)$
- $N = 3$ will require $\tilde{\beta} > 1 \Rightarrow$ serious sign problem
- truncation desirable: simpler worm simulation

worm method: start from

$$Z(u,v) = \int \left[\prod_z d\mu(s(z)) \right] \mathrm{e}^{\beta \sum_{l=\langle x\, y \rangle} s(x)\cdot s(y)}\, s(u)\cdot s(v)$$

and consider new ensemble

$$\mathcal{Z} = \sum_{u,v} \rho^{-1}(u-v) Z(u,v) \qquad [\rho > 0, \rho(0) = 1]$$

$\Longrightarrow$ relation:

$$\langle s(x)\cdot s(0)\rangle = \frac{Z(x,0)}{Z(0,0)} = \rho(x)\, \langle \delta_{x,u-v} \rangle_{\mathcal{Z}}$$

- $\rho(x)$ for noise optimization $\rightarrow$ later, imagine $\rho \equiv 1$

naive strong coupling expansion on each link $l = \langle x\, y \rangle$:

$$\mathrm{e}^{\beta s(x)\cdot s(y)} = \sum_{k(l)=0}^{\infty} \frac{\beta^{k(l)}}{k(l)!} [s(x)\cdot s(y)]^{k(l)}$$

- new link field $\{k(l)\}$, integrate $\{s(x)\} \rightarrow$ graphs

$\Longrightarrow$ graphs $\{\Lambda\}$:

- $k(l)$ lines on l, $\{u, v\}$ given by Λ

at each site we perform

$$\text{at } z: \quad \int d\mu(s)\, s^{a_1} s^{a_2} \cdots s^{a_d} \quad d(z) = \delta_{z,u} + \delta_{z,v} + \sum_{l, \partial l \ni z} k(l)$$

- $a_1, a_2, \ldots$. pairwise contracted in all possible ways, d even
- $\Rightarrow$closed loops and one string between u and $v \Rightarrow$ factors N
- loops unoriented, overlapping, intersecting

$$\mathcal{Z} = \sum_{\Lambda} N^{|\Lambda|}\, \beta^{\sum_l k(l)} \left[\prod_x 2^{-d(x)/2} \frac{\Gamma(N/2)}{\Gamma(N/2 + d(x)/2)} \right]$$

- weight allows for continuous N, positive for $N \in \mathbb{R}_+$

Worm update steps a la Prokof'ev Svistunov

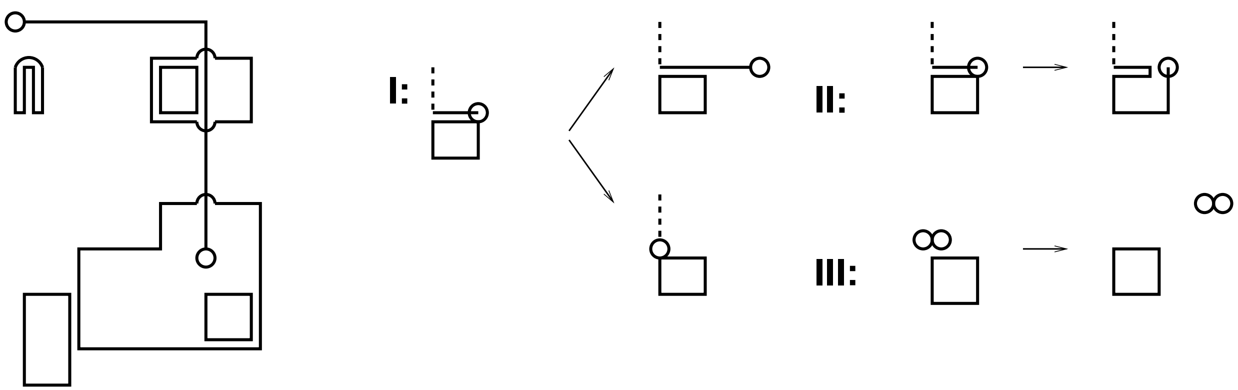
Metropolis proposals with suitable acc/rej:

I. hop: $u \to u' =$ nearest neighbor, adjust $(l = \langle u\, u' \rangle)$ $k_l \to k_l \pm 1$

II. re-connect lines around u

III. kick: if trivial string: relocate $u = v$ randomly

store configuration as linked list of pointers



- combine $\mathrm{O}(V)$ such steps to 'sweeps'
- critique: for II we need nonlocal information for $N^{|\Lambda|}$
- okay, even on large lattices, but...

Alternative representation

$$e^{\beta s(x)\cdot s(y)} = \sum_{k_1(l),k_2(l)\ldots k_N(l)=0}^{\infty} \left\{ \prod_a \frac{\beta^{k_a(l)}}{k_a(l)!} \left[s_a(x) s_a(y)\right]^{k_a(l)} \right\}$$

- loops and string each carry one of N colors
- $\Rightarrow$factor $N^{|\Lambda|}$ comes numerically (integer N only)
- store simple arrays, no re-connect steps
- strictly local steps now

Transition to Nienhuis:

in either form: restrict $k(l)$ [or$\sum_a k_a(l)$] to $\{0, 1\}$ only

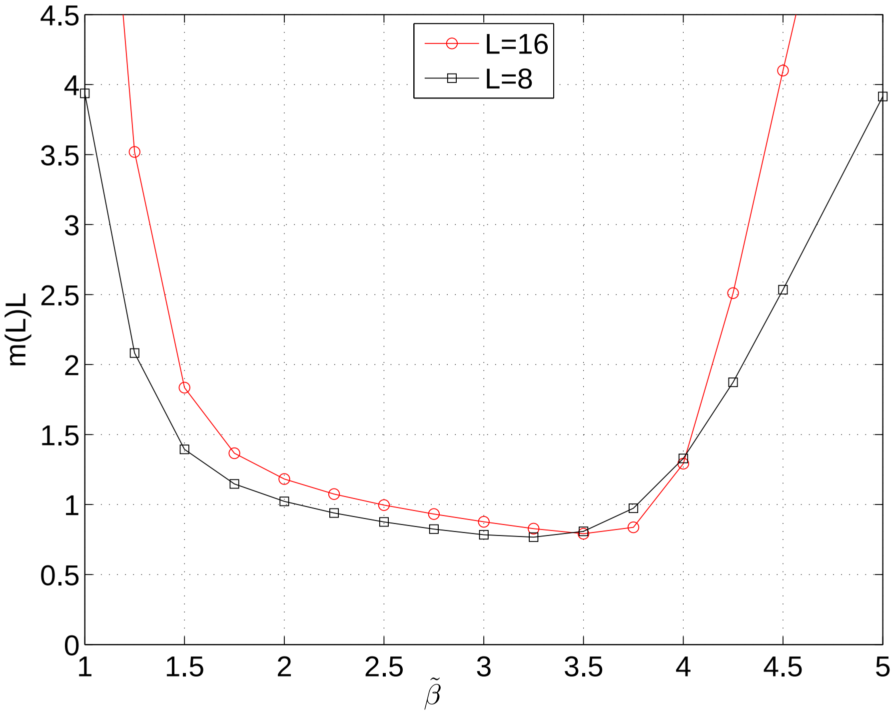
- lines in graphs overlap no more (but still cross)
- 2-d honeycomb lattice: $\Rightarrow$no crossing, disjoint loops (simpler)

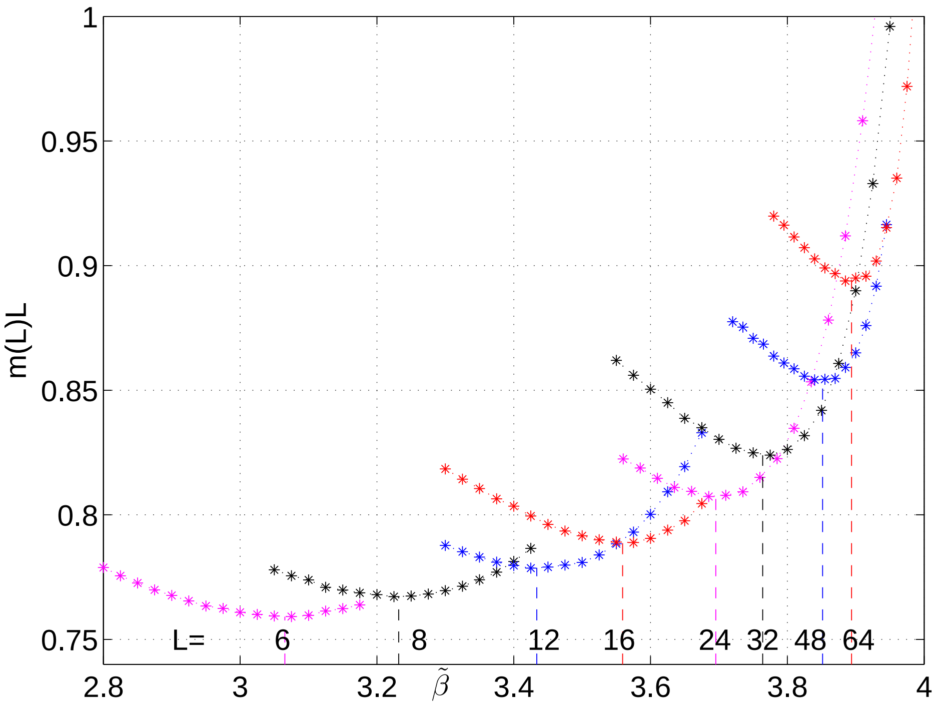
Step scaling function of the Nienhuis O(3) model, $D=2$

- mass gap $m(L)$, renormalized coupling $g^2(L)=m(L)L$
- SSF:

$$\Sigma(u,a/L)=g^2(2L)|_{g^2(L)=u}, \quad \sigma(u)=\lim_{a/L\to 0}\Sigma(u,a/L)$$

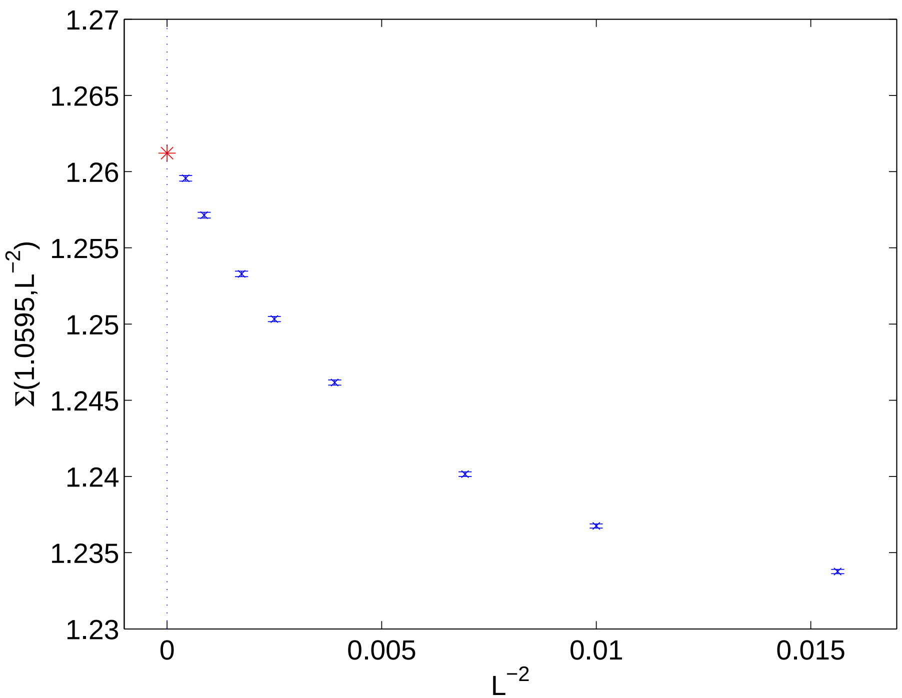
- $\sigma(u)$ universal, β function like, asymptotic freedom
- σ known exactly (Balog, Bethe Ansatz)
- lots of numerical tests, high precision
- measurable at 10^{-5} precision with worm methods
- endless plateaux of $m_{\rm eff}$ (use $\rho(x)$)
- no (practical) slowing down: cost $[\delta m/m=10^{-5}]\propto(LT/a^2)$

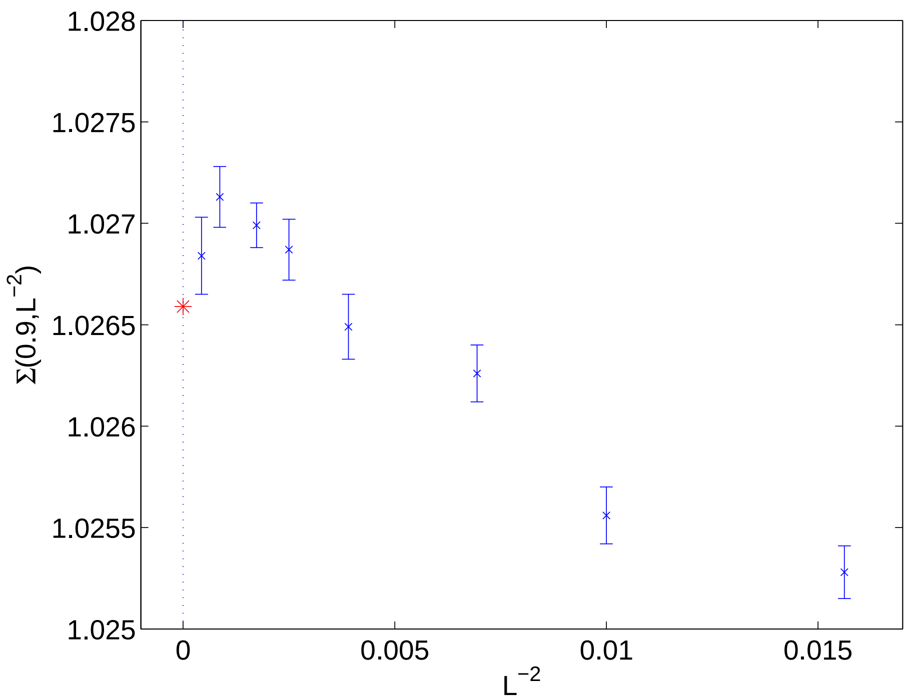




- only $g^2 > g^2_{\rm min}$ can be reached, growing with L
  - standard action, PT$\rightarrow$ $g^2 = 1/\beta + c(L)/\beta^2 + \mathrm{O}(\beta^{-3})$
  - Nienhuis: PT limit not reached in this renormalized g
- falling branches: $g^2(2L) > g^2(L)$, sign for as. freedom
- $\tilde\beta > 1$ clearly required $\rightarrow$ sign problem in $s(x)$, not $k_a(l)$

Measuring SSFs:





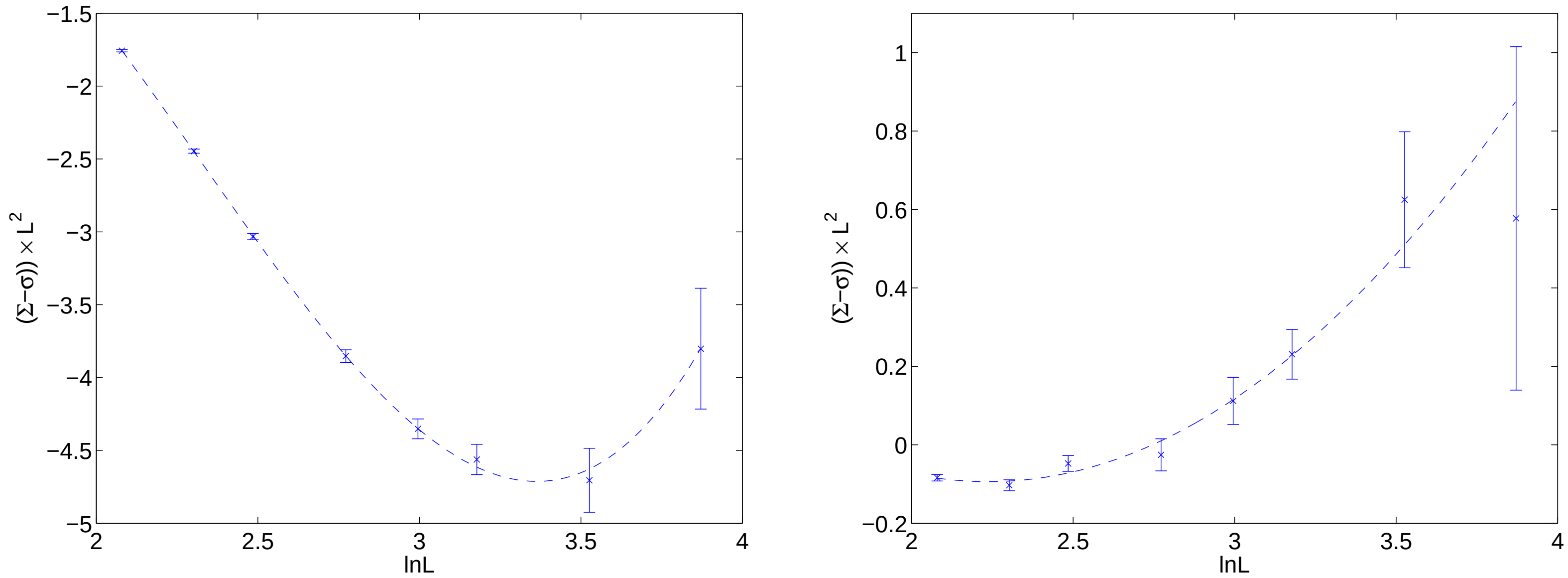
'traditional' $u = 1.0595$ $\qquad$ $u = 0.9$ close to min for $L = 48$

- up to $L = 48 \rightarrow 2L = 96$, $*$ = exact
- data seem to 'know' $*$, deviations small
- cut-off effects look non-monotonic at our precision

Balog, Niedermayer, Weisz:

$$\Sigma - \sigma = \frac{1}{L^2}[A \ln^3 L + B \ln^2 L + C \ln L + D]$$

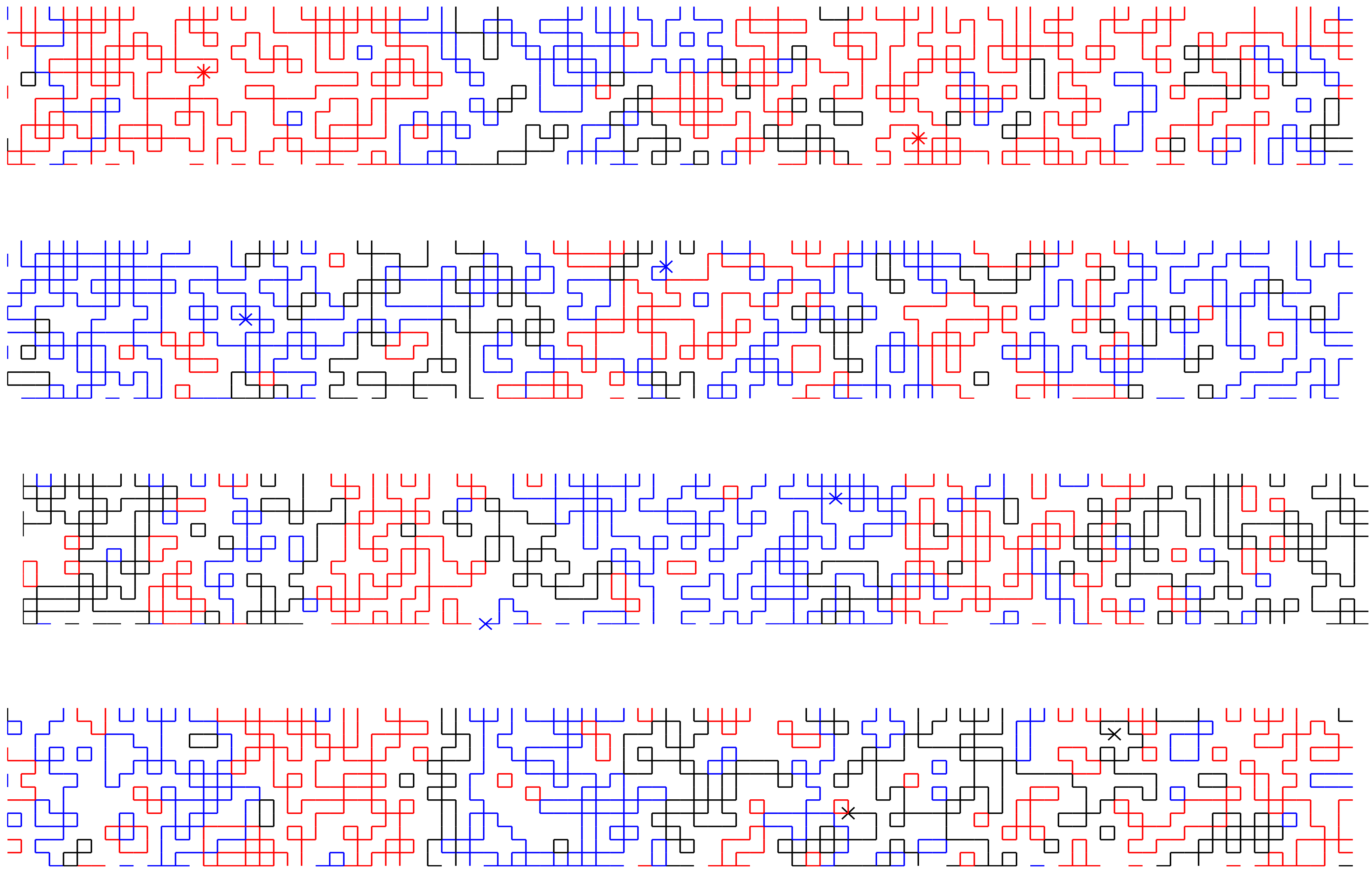
for a wide class of actions not including Nienhuis



$A = 0.97, B = -6.8, C = 13, D = -7.3$ $\quad$ $A = 0, B = 0.36, C = -1.6, D = 1.7$

- reasonable fits, competing logs

four configs: $N = 3$(red,blue,black), $L = 12$, $T = 96$, $\tilde{\beta} = 2.6804$, $u = 0.9$



Conclusions

- strange result
- for a given $\bar{g}^2$ only $(L/a) \leqslant c(\bar{g}^2)$ can be reached
  - $\rightarrow$ no complete continuum limit $a/L \rightarrow 0$
  - larger L/a possible for larger $\bar{g}^2$ $(c' > 0)$ (all $\bar{g}^2$??)
- may even be empty for too small $\bar{g}^2$
- BUT: for $\bar{g}^2$=1.0595, 0.9
  we got very close to exact answer up to tiny cutoff effects

- effective theory?
- analogue: triviality in 4d $\lambda\varphi^4$: for given $\lambda_R > 0$
  no complete limit $am_R \rightarrow 0$
- opposite: smaller λ_R allows smaller am_R