Faddeev calculation of kaonic deuterium

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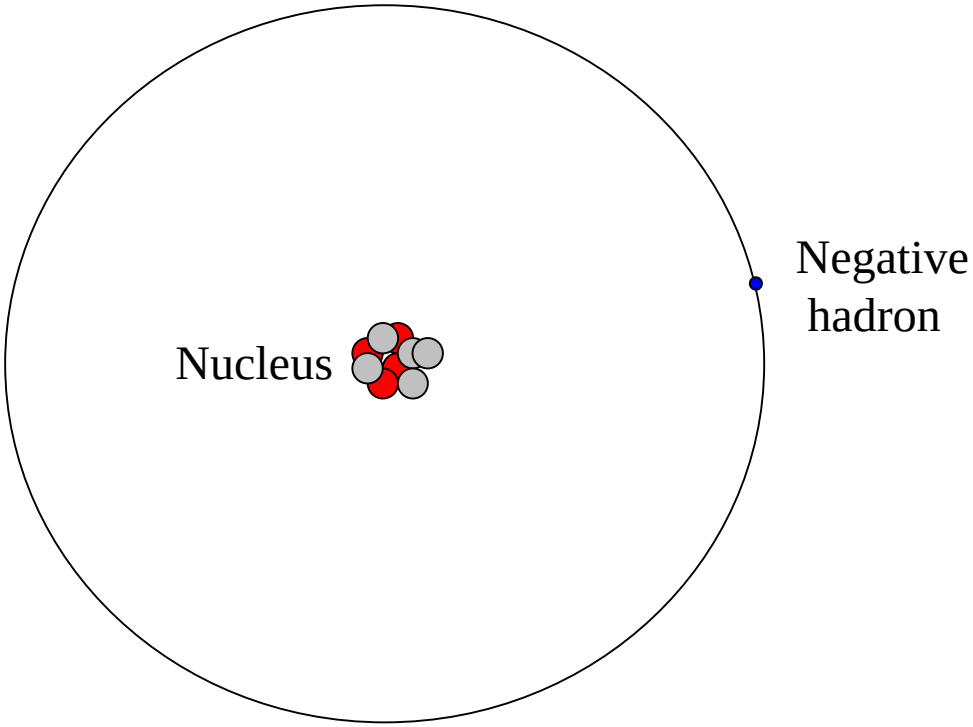


A.Deloff in „Fundamentals in hadronic atom theory”:
„.. the conventional picture of hadronic atoms (is) based on a two-body model Hamiltonian in which all strong interaction effects have been simulated by an absorptive potential representing the complicated interaction between the hadron and the nucleus...”

The simplest case to study deviation from two-body picture:

hadronic (kaonic) deuterium as a three-body system

Powerful methods to treat three-body problem:

(a) - Faddeev equations

(b) - variational methods (w.f. expansion in coordinate space)

but:

(a) - everlasting problem with the long range Coulomb force

(b) - two very different – and relevant - distance scales in hadronic atoms

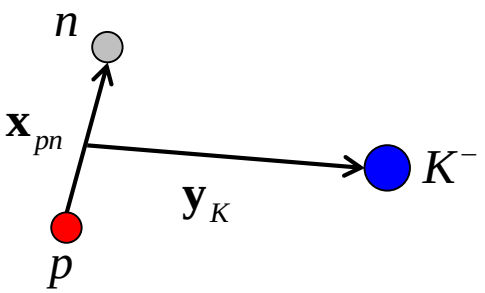
Some years ago Z. Papp proposed a method for simultaneous treatment of short range and Coulomb forces in three-body problem.

Z. Papp, W.Plessas, Phys. Rev. C54 (1996) 50

Basic idea: transform the Faddeev integral equations into matrix equations using a special discrete and complete set of the Coulomb Sturmian functions as a basis.

Successfully applied for short range + repulsive Coulomb forces (nuclear case) and purely Coulomb systems with attraction and repulsion.

Kaonic deuterium as a three-body system



Jacobi coordinates

$$(\mathbf{x}_{pn},\mathbf{y}_K),(\mathbf{x}_{pK},\mathbf{y}_n),(\mathbf{x}_{nK},\mathbf{y}_p)$$

Hamiltonian

$$H = H_0 + v^s_{np}(x_{np}) + v^s_{nK}(x_{nK}) + v^s_{pK}(x_{pK}) - \frac{e^2}{x_{pK}}$$

$$H_0 = -\frac{1}{2\mu_{pn}}\Delta_{\mathbf{x}_{pn}} - \frac{1}{2\mu_{pn,K}}\Delta_{\mathbf{y}_K} = h_0(x_{pn}) + h_0(\mathrm{y}_K)$$

$\alpha = (pn,K),(pK,n),(nK,p)$ partition channels

$\mathbf{x}_\alpha, \mathbf{y}_\alpha$ Jacobi coordinates

The first step

Faddeev equations are written in Noble form: the Coulomb interaction appears in the Green's functions

J.V. Noble, Phys.Rev. 161 (1967) 945

$$\Psi = \Psi_{np}(\mathbf{x}_{pn}, \mathbf{y}_K) + \Psi_{nK}(\mathbf{x}_{nK}, \mathbf{y}_p) + \Psi_{pK}(\mathbf{x}_{pK}, \mathbf{y}_n)$$

$$\Psi_{np} = \left(z - H_0 - v^s_{np}(x_{np}) + \frac{e^2}{|c_{np}\mathbf{x}_{np} + \mathbf{y}_K|} \right)^{-1} v^s_{np}(x_{np})(\Psi_{nK} + \Psi_{pK})$$

$$\Psi_{nK} = \left(z - H_0 - v^s_{nK}(x_{nK}) + \frac{e^2}{|c_{nK}\mathbf{x}_{nK} + \mathbf{y}_p|} \right)^{-1} v^s_{nK}(x_{nK})(\Psi_{np} + \Psi_{pK})$$

$$\Psi_{pK} = \left(z - H_0 - v^s_{pK}(x_{pK}) + \frac{e^2}{x_{pK}} \right)^{-1} v^s_{pK}(x_{pK})(\Psi_{np} + \Psi_{nK})$$

c – mass coefficients

The Coulomb interaction is the same in all three equations, only expressed in different coordinates

Shorter notation: $\Psi = \sum_\alpha \Psi_\alpha \quad ; \qquad \Psi_\alpha = G_\alpha(z) v_\alpha^s \sum_{\gamma \neq \alpha} \Psi_\gamma$

$$G_\alpha(z) = \left(z - H_0 - v_\alpha^s + \frac{e^2}{x_{pK}} \right)^{-1}$$

The second step

Obtain matrix equation by inserting (approximate) unit operators using a double Coulomb Sturmian basis

Coulomb Sturmian functions

$\langle \mathbf{r} \mid nlm \rangle = \langle \mathbf{r} \mid i \rangle = N_{nl} r^l e^{-br} L_n^{2l+1}(2br) Y_{lm}(\hat{\mathbf{r}})$ $\quad b$ - range parameter

$\langle i \mid \frac{1}{r} \mid j \rangle = \delta_{ij} \qquad \langle \mathbf{r} \mid \tilde{i} \rangle = \langle \mathbf{r} \mid \frac{1}{r} \mid j \rangle \qquad \langle i \mid \tilde{j} \rangle = \langle \tilde{i} \mid j \rangle = \delta_{ij}$ biorthogonal set

$$\sum_{i=0}^{\infty} | i \rangle \langle \tilde{i} | = \sum_{i=0}^{\infty} | \tilde{i} \rangle \langle i | = \hat{1} \approx \sum_{i=0}^{N} | i \rangle \langle \tilde{i} | \qquad i = nlm$$

Matrix elements of a two-body Hamiltonian with Coulomb potential:

$$h_c = -\frac{1}{2m}\Delta_{\mathbf{r}} \pm \frac{e^2}{r}$$

$$\langle i \,|\, z - h_c \,|\, i' \rangle = \frac{1}{2b}\delta_{ll'}\left\{\begin{array}{l} -\delta_{n,n'+1}\left[\sqrt{n(n+2l+1)}\,(z + b^2/2m)\right] \\ +\delta_{n,n'}\left[2(n+l+1)(z - b^2/2m) \mp 2be^2\right] \\ -\delta_{n,n'-1}\left[\sqrt{(n+1)(n+2l+2)}\,(z + b^2/2m)\right] \end{array}\right\}$$

$$\langle i \,|\, (z - h_c) g_c(z) \,|\, \tilde{j} \rangle = \delta_{ij} = \sum_{s=0}^{\infty} \langle i \,|\, (z - h_c) \,|\, s \rangle \langle \tilde{s} \,|\, g_c(z) \,|\, \tilde{j} \rangle$$

- infinite tridiagonal set of equations for the matrix elements $\langle \tilde{i} \,|\, g_c(z) \,|\, \tilde{j} \rangle$

Can be solved exactly.

The same holds for the matrix elements of the free Green's function $\langle \tilde{i} \,|\, g_0(z) \,|\, \tilde{j} \rangle$

Using $\hat{1}_\alpha = \sum_{\mu}^{N_\alpha} |\mu\rangle_\alpha \, {}_\alpha\langle\tilde{\mu}| \approx \hat{1}$

$$\langle \mathbf{x}_\alpha, \mathbf{y}_\alpha \,|\, \mu\rangle_\alpha = \langle \mathbf{x}_\alpha \,|\, i\rangle\langle \mathbf{y}_\alpha \,|\, I\rangle = \langle \mathbf{x}_\alpha \,|\, nl\rangle\langle \mathbf{y}_\alpha \,|\, NL\rangle;\ \mu = iI = nlNL$$

we get the equations

$$X^{\alpha}_{\mu} = \sum_{(\mu)} [G_\alpha(z)]_{\mu\mu'} (v^s_\alpha)_{\mu'\mu''} \sum_{\gamma\neq\alpha} (M^{\alpha\gamma})_{\mu''\mu'''} X^{\gamma}_{\mu'''}$$

with unknowns $X^{\alpha}_{\mu} = {}_\alpha\langle\tilde{\mu}\,|\,\Psi_\alpha\rangle$

and overlap matrix $(M^{\alpha\gamma})_{\mu\mu'} = {}_\alpha\langle\tilde{\mu}\,|\,\mu'\rangle_\gamma$

Eigenvalue equation to be solved $Det(z) = 0$

$(v^s_\alpha)_{\mu\mu'} = {}_\alpha\langle\mu\,|\,v^s_\alpha\,|\,\mu'\rangle_\alpha$ and $(M^{\alpha\gamma})_{\mu\mu'}$ can be calculated numerically

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The basic quantities of Papp's method are the matrix elements of the Green's functions:

$$[G_\alpha(z)]_{\mu\mu'} = {}_\alpha\langle\tilde{\mu}\,|\,(z - h_0(x_\alpha) - h_0(y_\alpha) - v^s_\alpha(x_\alpha) - \frac{e^2}{x_{pK}})^{-1}\,|\,\tilde{\mu}'\rangle_\alpha$$

For $\alpha = (pK, n)$ $G_\alpha(z)$ is the Green's function of two non-interacting subsystems and can be calculated by a convolution integral along a suitable contour in the complex energy plane:

$$G_\alpha(z) = \oint g_\alpha^{sc}(\varepsilon; x_\alpha)\, g_\alpha^{0}(z-\varepsilon; y_\alpha)\, d\varepsilon$$

with

$$g_\alpha^{sc}(z; x_\alpha) = (z - h_0(x_\alpha) - v_\alpha^s(x_\alpha) + \frac{e^2}{x_\alpha})^{-1}$$

$$g_\alpha^{0}(z; y_\alpha) = (z - h_0(y_\alpha))^{-1}$$

$[g_\alpha^{0}(z; y_\alpha)]_{II'}$ is known

$[g_\alpha^{sc}(z; x_\alpha)]_{ii'}$ calculated from the matrix equation

$$[g_\alpha^{sc}(z; x_\alpha)]_{ii'} = [g_\alpha^{c}(z; x_\alpha)]_{ii'} + \sum_{i'', i'''} [g_\alpha^{c}(z; x_\alpha)]_{ii''} (v_\alpha^s)_{i''i'''} [g_\alpha^{sc}(z; x_\alpha)]_{i'''i'}$$

For $\alpha \neq (pK, n)$ an intermediate step is required:

$$-\frac{e^2}{|c_\alpha \mathbf{x}_\alpha + \mathbf{y}_\alpha|} = -\frac{e^2}{y_\alpha} + \left(\frac{e^2}{y_\alpha} - \frac{e^2}{|c_\alpha \mathbf{x}_\alpha + \mathbf{y}_\alpha|} \right) = V_\alpha^{c,ch}(y_\alpha) + U_\alpha(\mathbf{x}_\alpha, \mathbf{y}_\alpha)$$

with $V_\alpha^{c,ch}(y_\alpha)$ - channel Coulomb interaction

$U_\alpha(\mathbf{x}_\alpha, \mathbf{y}_\alpha)$ - polarization potential

$$G_\alpha(z) = G_\alpha^{ch}(z) + G_\alpha^{ch}(z) U_\alpha G_\alpha(z)$$

$$G_\alpha^{ch}(z) = (z - h_0(x_\alpha) - h_0(y_\alpha) - v_\alpha^s(x_\alpha) + \frac{e^2}{y_\alpha})^{-1}$$ - channel Green's function,

again of non-interacting subsystems $\rightarrow$ convolution integral

$$G_\alpha^{ch}(z) = \int g_\alpha^s(z; x_\alpha)\, g_\alpha^{c,ch}(z - \varepsilon; y_\alpha)\, d\varepsilon$$

$(U_\alpha)_{\mu\mu'} = {}_\alpha\langle \mu | U_\alpha | \mu' \rangle_\alpha$ - numerical integration

A few remarks about the actual calculation

(l) There is a "dominant" channel Green's function:

$$G^{ch}_{(pn,K)}(z) = \left(z - h_0(x_{np}) - h_0(y_K) - v^s_{np}(x_{np}) + \frac{e^2}{y_K} \right)^{-1},$$

which corresponds to a deuteron and a kaon "feeling" a Coulomb force from the c.m. of the deuteron. Its lowest eigenvalue $z_0 = E_d + E^{ch}_{1s}$ is the reference point, from which the energy shift is measured

$$\Delta z = z - z_0$$

At $z = z_0$ all m.e.-s of $G^{ch}_{(pn,K)}(z)$ are singular, the search for $Det(z) = 0$ is performed in the vicinity of z_0.

(II) Even in the absence of the strong interaction of the kaon with the nucleons, the presence of the polarization potential $U_{(pn,K)}$ causes a certain (real) shift of the eigenvalue from z_0 to z_1. In principle, the strong shift should be measured from z_1 instead of z_0. However, the effect is small, in our case

$$z_1 - z_0 \approx 10 \ eV$$

(III) The $\bar{K}N$ interaction is isospin conserving and acts in $I=0$ and $I=1$ states, therefore in our "particle" representation the Faddeev component $\Psi_{pK,n}$ is a column vector:

$$\Psi_{(pK,n)} = \begin{pmatrix} \Psi_{(pK^-,n)} \\ \Psi_{(nK^0,n)} \end{pmatrix}$$

while the v^s_{pK} interaction is a 2x2 matrix:

$$v^s_{pK} = \begin{pmatrix} v^s_{pK^-,pK^-} & v^s_{pK^-,nK^0} \\ v^s_{nK^0,pK^-} & v^s_{nK^0,nK^0} \end{pmatrix}$$

(IV) A rather heavy numerical work with a lot of small but important technical details should be done.

The convergence of the method depends on the good choice of the range parameters b of the Sturmian functions:

- different in different partition channels,
- different for the x and y variables.

For a good choice of b 30-40 functions for one variable give an accuracy of ~0 .5% (~ 1-2 eV).

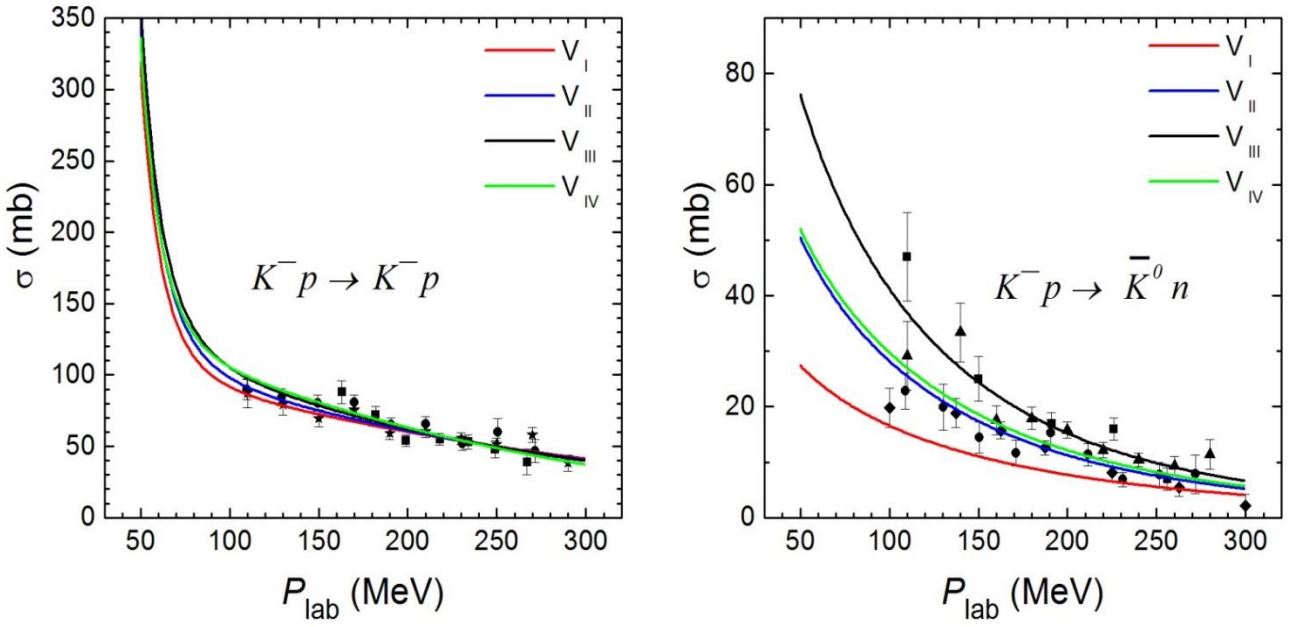
The dimension of the final matrix for 40 functions in each variable is 4800.

Input

Four models of $\bar{K}N$ interaction:

one-term separable complex potentials with Yamaguchi form factors,

reproduce: scattering lengths and/or $\Lambda(1405)$ positions of coupled-channel potentials, give 1s level shifts in kaonic hydrogen within or close to the SIDDHARTA value. Their $K^- p$ cross sections:



The choice of the potentials is not the main issue of this calculation

Comparison with approximations

The exact (accurate to 1 *eV*) results are compared with the commonly used approximations. All of them used results of a three-body Faddeev calculation of the system without Coulomb interaction.

Deser and corrected Deser: K^-d scattering length was used

Optical potential: fitted to the low energy elastic K^-d amplitudes (including scattering length and effective range),

Lippmann-Schwinger equation with the optical plus Coulomb potential was solved

N.V. Shevchenko, Nucl. Phys. A890-891 (2012) 50

Results: *1s* level shift ΔE and width $\Gamma = 2\,Im(\Delta E)$ *(eV)* of kaonic deuterium

	Kaonic hydrogen shift	Kaonic deuterium shift			
		Deser	corrected Deser	Optical potential	Exact
V_I	-280 + 268 i				-641 + 428 i
V_{II}	-217 + 292 i				-646 + 444 i
V_{III}	-219 + 293 i				-732 + 490 i
V_{IV}	-280 + 266 i				-736 + 413 i

Results: *1s* level shift ΔE and width $\Gamma = 2\,Im(\Delta E)$ *(eV)* of kaonic deuterium

	Kaonic hydrogen shift	Kaonic deuterium shift			
		Deser	corrected Deser	Optical potential	Exact
V_I	-280 + 268 i	-723 + 596 i			-641 + 428 i
V_{II}	-217 + 292 i	-732 + 634 i			-646 + 444 i
V_{III}	-219 + 293 i	-837 + 744 i			-732 + 490 i
V_{IV}	-280 + 266 i	-854 + 604 i			-736 + 413 i

Results: *1s* level shift ΔE and width $\Gamma = 2\,Im(\Delta E)$ *(eV)* of kaonic deuterium

	Kaonic hydrogen shift	Kaonic deuterium shift			
		Deser	corrected Deser	Optical potential	Exact
V_I	-280 + 268 i	-723 + 596 i	-675 + 351 i		-641 + 428 i
V_{II}	-217 + 292 i	-732 + 634 i	-694 + 370 i		-646 + 444 i
V_{III}	-219 + 293 i	-837 + 744 i	-795 + 390 i		-732 + 490 i
V_{IV}	-280 + 266 i	-854 + 604 i	-750 + 310 i		-736 + 413 i

Results: *1s* level shift ΔE and width $\Gamma = 2\,Im(\Delta E)$ *(eV)* of kaonic deuterium

	Kaonic hydrogen shift	Kaonic deuterium shift			
		Deser	corrected Deser	Optical potential	Exact
V_I	-280 + 268 i	-723 + 596 i	-675 + 351 i	-650 + 434 i	-641 + 428 i
V_{II}	-217 + 292 i	-732 + 634 i	-694 + 370 i	-658 + 460 i	-646 + 444 i
V_{III}	-219 + 293 i	-837 + 744 i	-795 + 390 i	-747 + 517 i	-732 + 490 i
V_{IV}	-280 + 266 i	-854 + 604 i	-750 + 310 i	-740 + 422 i	-736 + 413 i

Conclusions

The first (test) exact calculation of the *1s* level shift in kaonic deuterium was performed

Approximate calculations	Their accuracy
Simple Deser	rough estimation
Corrected Deser	~10% error in real part, ~25% error in imaginary part
Optical potential	good approximation, as assumed (<= 2% error in Re, <=5% in Im)

The next step

Extend the method for energy-dependent $\bar{K}N$ interactions in order to

- incorporate chirally motivated potentials with inherent energy dependence
- properly account for the $\pi\Sigma$ channel via an energy dependent optical potential