GLUON PROPAGATOR IN AN EXTERNAL FIELD: WHAT HAPPENS WHEN THE FIELD IS REMOVED?

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INTRODUCTION

To relieve the tension my question has provoked, and in case I run out of time, I will answer it now:

NOTHING

More precisely. We are in minimal Landau gauge, so all configurations are transverse $\partial_\mu A_\mu = 0$, and lie inside the Gribov region Ω.

We introduce a source

$$(J, A) \tag{1}$$

into the action, where

$$J^a_\mu(x) = h \cos(kx_1)\ \delta^{a3}\ \delta_{\mu 2}, \tag{2}$$

which is chosen to be transverse, $\partial_\mu J^a_\mu = 0$. Here h is the strength of the external field and is the analog of an external magnetic field applied to a spin system, which is modulated by $\cos(kx_1)$ It turns out to be a valuable probe.

The gluon propagator of momentum k in the presence of this source, $D(k, h)$ is the second derivative of the free energy

$$D(k, h) = \frac{1}{V} \frac{\partial^2 W(k, h)}{\partial h^2} \tag{3}$$

We are interested in the infrared limit $k \to 0$ of the gluon propagator $D(k)$. It has been studied numerically in Euclidean $d = 2, 3, 4$ dimensions at $h = 0$. In our notation, $D(k) = D(k, h = 0)$. Lattice data, taken at $h = 0$, indicate that in $d = 3, 4$ Euclidean dimensions, the gluon propagator $D(k, 0)$ is finite in the infrared,

$$\lim_{k \to 0} D(k, 0) > 0. \tag{4}$$

To be consistent with this, lattice data taken at finite h must satisfy

$$\lim_{k \to 0} \lim_{h \to 0} D(k, h) > 0. \tag{5}$$

On the other hand, it has been proven that at finite external field $h > 0$ the gluon propagator vanishes in the infrared,

$$\lim_{k \to 0} D(k, h) = 0 \qquad \text{for all } h > 0. \tag{6}$$

and since the limit of 0 is 0, we have

$$\lim_{h \to 0} \lim_{k \to 0} D(k, h) = 0. \tag{7}$$

and indeed, nothing happens in the limit $h \to 0$, as advertised.

If we take the lattice data at face value in $d = 3$ and 4 dimensions, the order of limits does not commute, and $D(k, h)$ is not analytic in h at $k = 0$!

I now present the result of our numerical studies.

NUMERI CAL STUDY OF FREE ENERGY AND GLUON PROPAGATOR IN AN EXTERNAL FIELD

The free energy is defined by

$$\exp[W(J)] = \langle \exp[(J, A)] \rangle \tag{8}$$

The expectation-value is calculated by Monte Carlo. Gauge fixing is done by minimization to minimal Landau gauge, so only the inside of the first Gribov region is sampled,

$$A \in \Omega; \tag{9}$$

$$\partial \cdot A = 0; \qquad \text{and} \qquad \lambda_0(A) \geq 0. \tag{10}$$

Figure 1 is a log-log plot, of $W(k, h)$ vs h at various fixed k. There are two regions separated by a critical value $h_{\rm cr}$ of the source

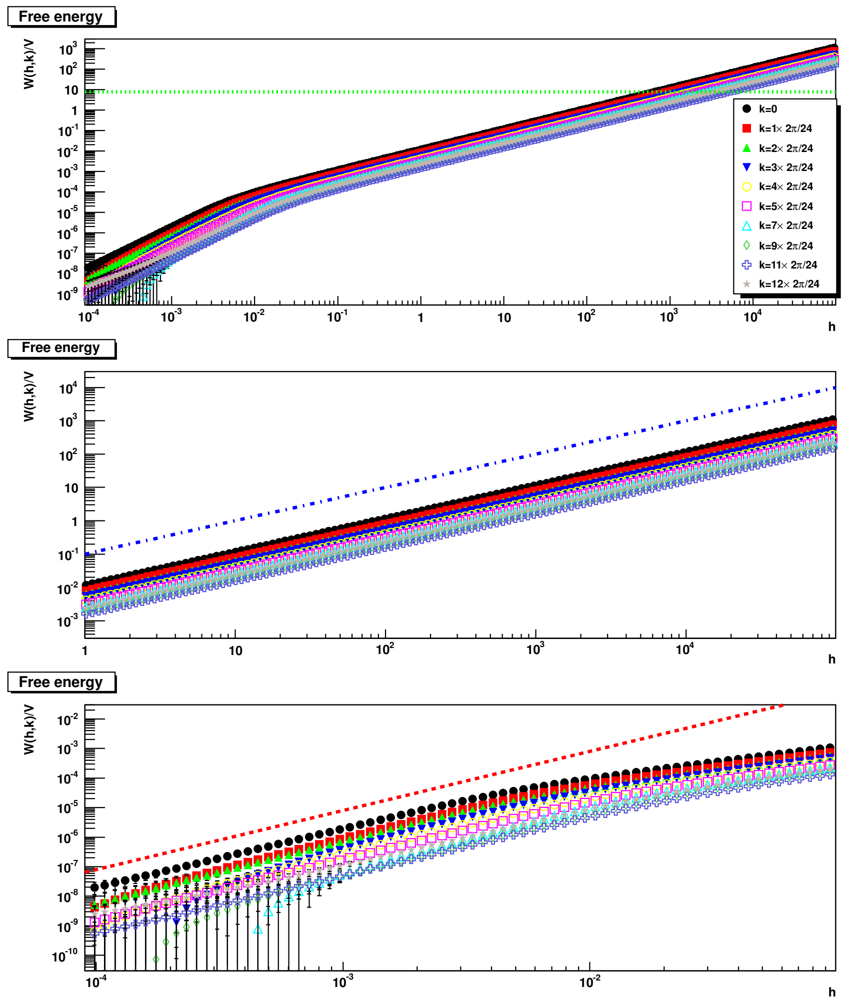


Figure 1: The free energy density $W(h,k)/V$ in lattice units as a function of h for various vales of k. The middle and bottom panels show magnifications of the region at large and small h, respectively. The value of h is limited by (**??**), indicated by the green dashed line in the top panel. In the middle panel the blue dashed line is linear in h, while the red dashed line the bottom panel is proportional to h^2. The four-dimensional 24^4 lattice has at $\beta = 2.221/a = 0.2$ fm a physical volume of $(4.8\ \text{fm})^4$.

strength h. Straight lines are power laws in h. The slope gives the power. For $h > h_{\rm cr}$, the slope is 1 and, so $W(k,h)$ rises linearly with h. This is consistent with the proven bound

$$W(k,h) \leq kh \text{ for } h > h_{\rm cr}. \qquad (11)$$

We are seeing the predicted asymptotic linear behavior. At small h the slope is 2, corresponding to a quadratic h^2 dependence. Recall that the free energy $W(k,h)$ is the generating functional of the connected correlators. The second derivative of the free energy is the gluon propagator,

$$\frac{1}{V}\frac{\partial^2 W(k,h)}{\partial h^2}|_{h=0} = D(k,0). \qquad (12)$$

At low h we are seeing the leading term in the power series expansion

$$W(k,h) = (1/2)D(k,0)h^2 + ... \text{ for } h < h_{\rm cr} \qquad (13)$$

Thus, the free energy has a very simple behavior. Below $h_{\rm cr}$ we

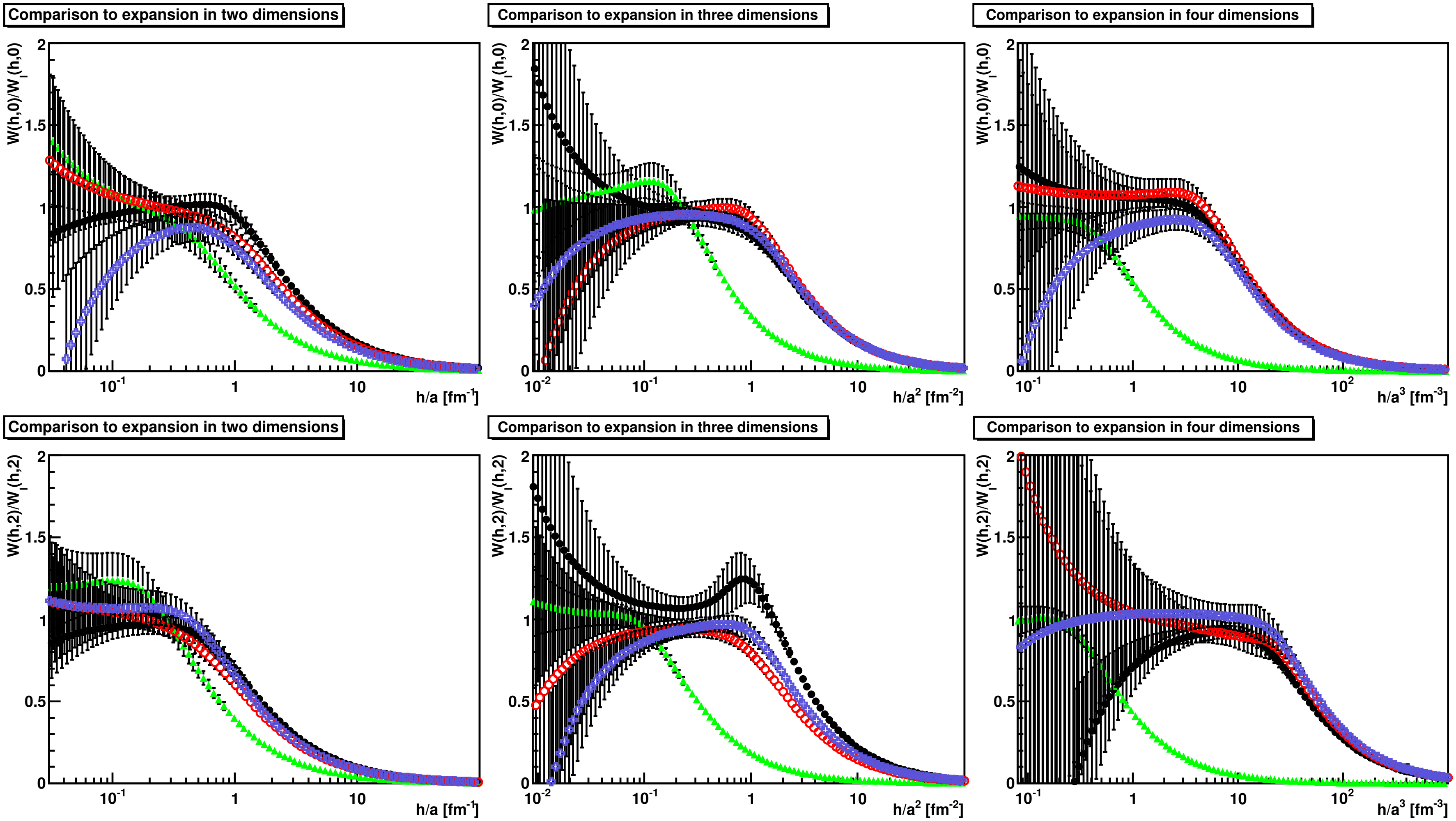


Figure 2: The ratio of the free energy, divided by the leading Taylor coefficient as a function of the dimensionful field h. The fixed volume is $(12\ \text{fm})^2$ in two dimensions, $(3.6\ \text{fm})^3$ in three dimensions and $(1.2\ \text{fm})^4$ in four dimensions. The largest volume can be taken from table **??**. Results are shown for $2\pi Lk = 0, 1, 2, 3$ from top to bottom.

see the h^2 term in the power series expansion, and above $h_{\rm cr}$, $W(k,h)$ is linear in h. We see this behavior in the lattice data, Figure 2. Below $h_{\rm cr}$ the ratio of $W(k,h)/(1/2)D(k,0)h^2$ is unity. A revealing feature of this figure is the green curve that represents a larger volume V than the others. At larger volume the region where $a(k,h)$ is constant moves toward the origin $h=0$.

What is going on? Why is the asymptotic behavior linear?

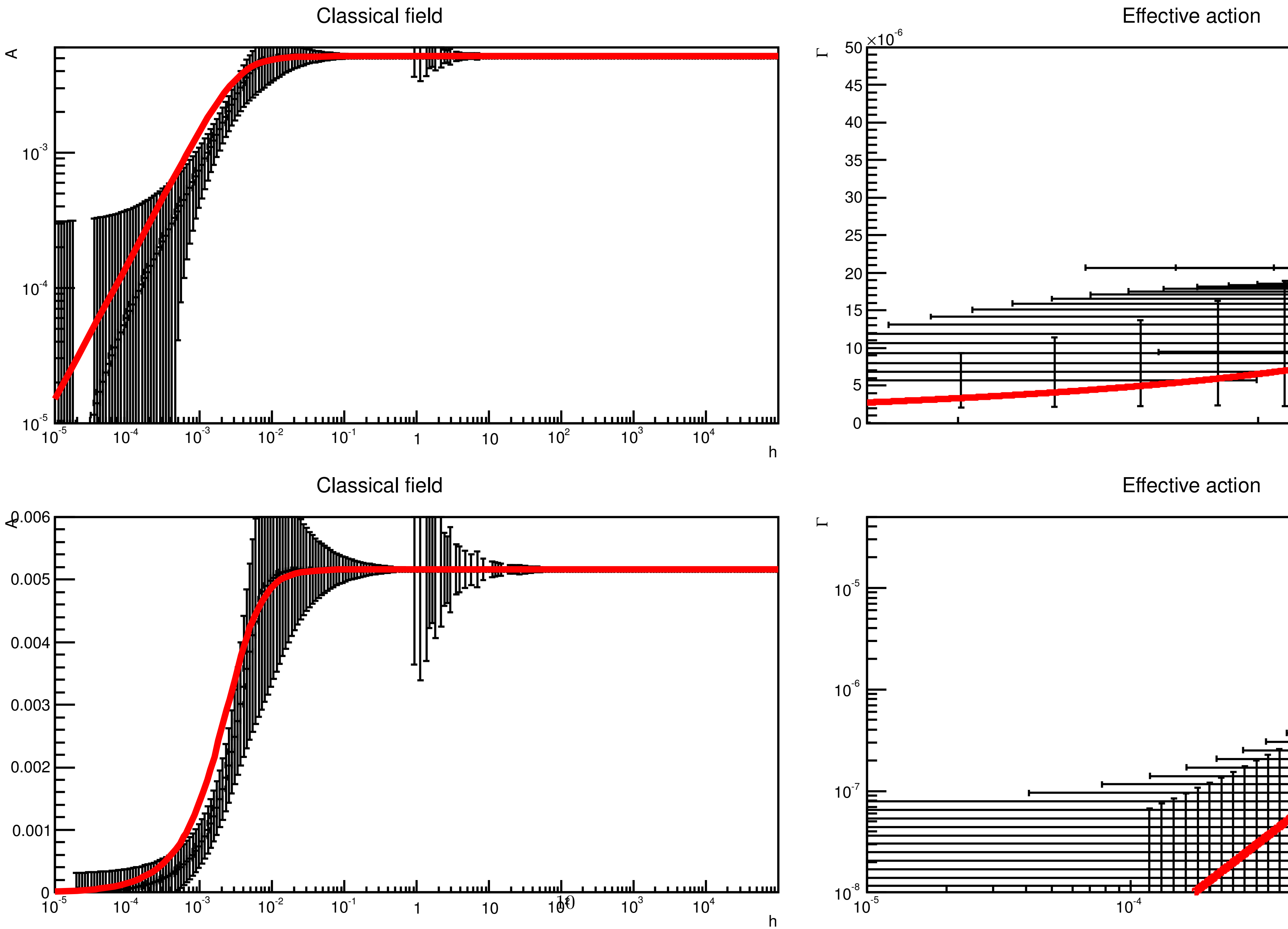
Let us look at the derivative,

$$a(k,h) = \frac{1}{V}\frac{\partial W(k,h)}{\partial h}, \tag{14}$$

which is the expectation-value of the k-th fourier component of the gluon field $A(x)$ in the presence of the external source $h\cos(kx_1)\delta_{\mu 2}\delta^{a3}$.

$$a(k,h) = \langle a(k)\rangle_h. \tag{15}$$

Figure 3, upper left panel shows a log-log plot of $a(k,h)$ as a


Classical field
A
h
Effective action
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Classical field
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function of the external field h at fixed k. The two regions are again visible. For $h > h_{\rm cr}$, $a(k,h)$ is constant, no matter how strong h is. corresponding to,

$$a(k,h) = ck \text{ for } h > h_{\rm cr}. \tag{16}$$

At low source strength a reasonable fit is

$$a(k,h) = D(k,0)h \text{ for } h < h_{\rm cr}. \tag{17}$$

Why is $a(k,h)$ constant above a critical field $h > h_{\rm cr}$ NO MATTER HOW STRONG?

ANSWER: For large h the external field pushes $a(k,h)$ against the Gribov horizon where it stops of necessity.

Where is the Gribov horizon?

Figure 4 shows the relation of the source J and the configuration $A^* = A^*(J)$ which maximizes the inner product (J, A) for fixed J and all A in the Gribov region.

Here we consider configurations in the plane labelled by two parameters, b and c,

$$A^a_\mu(x; b, c) = [b + c\cos(kx_1)]\delta_{\mu 2}\delta^{a3}. \tag{18}$$

The Gribov horizon is found by calculating the lowest non-trivial eigenvalue $\lambda_0(A)$ of the Faddeev-Popov operator $M(A) = -\partial^2 - A_\mu \times \partial_\mu$, for $A(b, c)$ of this form, on a Euclidean volume $V = L^d$. The maximum value of b is $b = 2\pi/L$, and the maximum value of c is $c = k/\sqrt{2}$. This calculation is for large volumes L^d at fixed

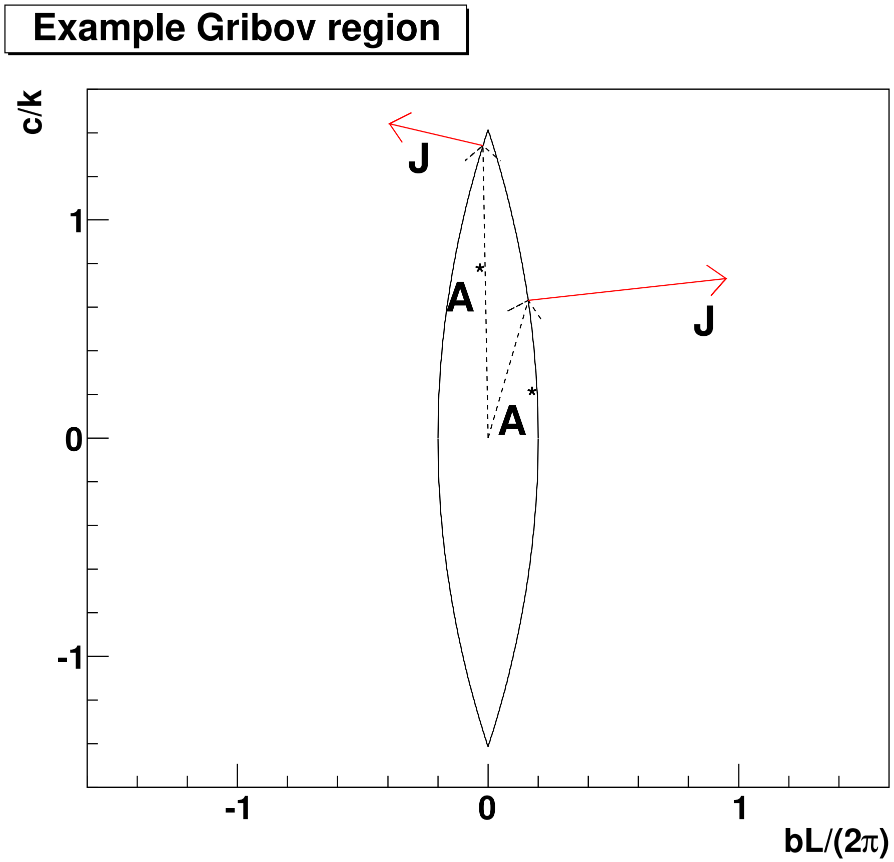

Figure 4: The intersection of the Gribov region Ω with the 2-plane $P(b,c)$ is contained between the two parabolas. For each J there is a bound on the free energy given by $W(J) \leq (J, A^*)$, where J and $A^* = A^*(J)$ are illustrated here.

momentum k, so $2\pi/L << k$. The source J is in the direction of the normal to the Gribov horizon,

$$J_x = \frac{\partial \lambda_0(A)}{\partial A_x}, \tag{19}$$

which is calculated from

$$J^a(x) = -f^{cab}\psi_0^{c*}(x)\psi_0^b(x). \tag{20}$$

WHERE DO THE BOUNDS COME FROM?

They all follow from a very simple and general bound,

$$\begin{aligned} \exp W(J) &= \int_\Omega dA\ \rho(A) \exp(J, A) \\ &\leq \int_\Omega dA\ \rho(A) \exp(J, A^*) = \exp(J, A^*), \end{aligned} \tag{21}$$

where A^* maximizes (J, A) for all A in Ω,

$$(J, A^*) = \max(J, A) \text{ for all } A \in \Omega. \tag{22}$$

It is easy to show that A^* lies on the boundary of Ω,

$$A^* \in \partial\Omega. \tag{23}$$

The asymptotic form of the free energy is given by

$$W_{\rm as}(J) \equiv (J, A^*(J)), \tag{24}$$

It is linear in J because

$$\max(tJ, A) = t \max(J, A) \tag{25}$$

for $t > 0$, and we have the bound

$$W(J) \leq W_{\rm as}(J). \tag{26}$$

Because $W_{\rm as}(J)$ is linear in J, this explains the linear behavior observed in fig. 1.

For a source of the form $J_\mu^a(x) = h\sqrt{2}\cos(kx_1)\delta_{\mu,2}\delta^{a,3}$, one can actually calculate A^*, the point on the Gribov horizon, and the bound is given by

$$\begin{aligned} w(0,h) &\leq |h|(2\pi/L) \\ w(k,h) &\leq |h|k \qquad \text{for } k >> 2\pi/L. \end{aligned} \tag{27}$$

This leads to the bound on

$$a(k,h) = \langle a_k \rangle_h = \frac{1}{V}\frac{\partial W(k,h)}{\partial h} \tag{28}$$

given by

$$0 \leq a(k,h) \leq a(k,\infty) \leq k. \tag{29}$$

This expresses the proximity of the Gribov horizon in infrared directions. It also shows that in the infrared limit no colored field can be excited.

From

$$D(k,h) = \frac{\partial a(k,h)}{\partial h} \tag{30}$$

we get the bound on the gluon propagator

$$\int_0^\infty dh \; D(k,h) \leq k. \tag{31}$$

This gives

$$\lim_{k\to 0}\int_0^\infty dh\ D(k,h) = 0. \tag{32}$$

Because $D(k,h)$ is a variance, it is positive, $D(k,h) \geq 0$, and we conclude

$$\lim_{k\to 0} D(k,h) = 0. \tag{33}$$

This implies

$$\lim_{h\to 0}\lim_{k\to 0} D(k,h) = 0. \tag{34}$$

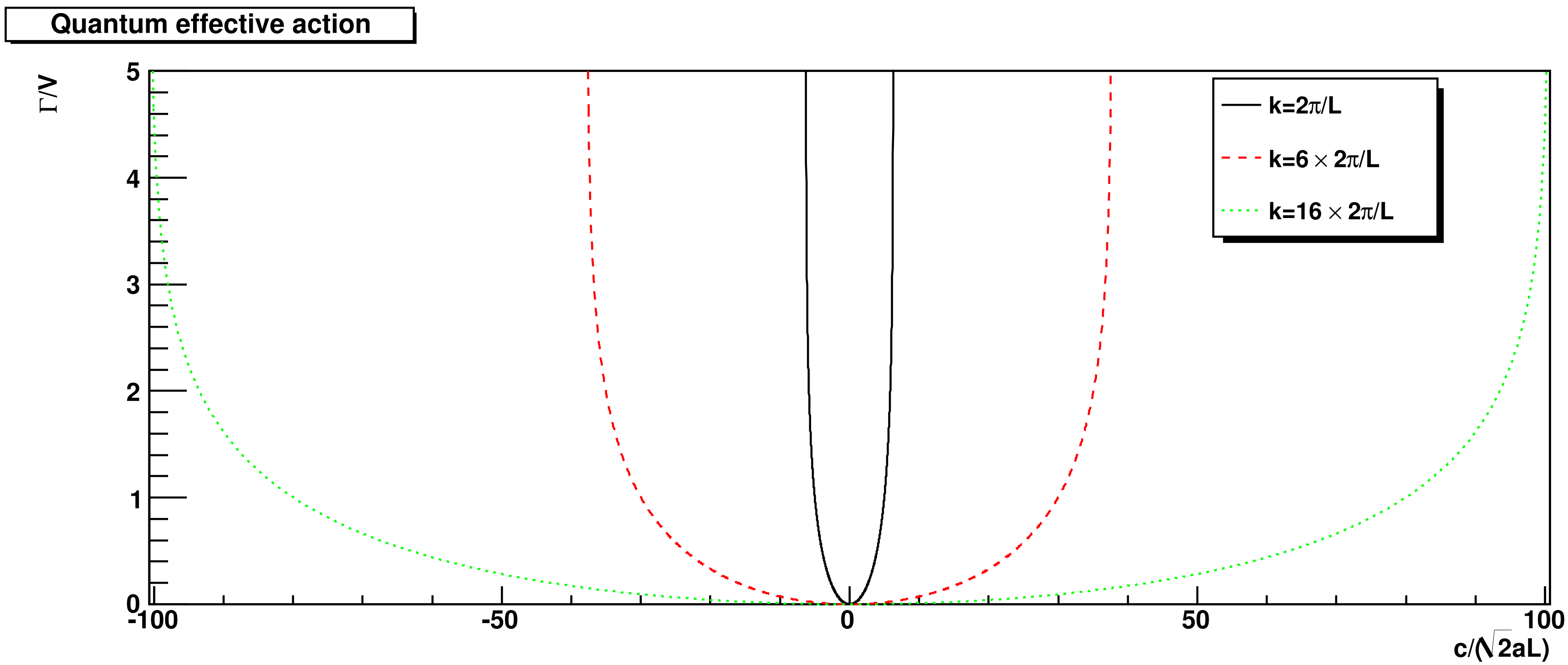

Figure 5: The quantum effective action (??) for $\hat{\gamma} = 1$. The lattice spacing is denoted by a.

An excellent fit to the data is provided by

$$W(k,h)/V = [\hat{\gamma}^2(k) + c^2k^2h^2]^{1/2} - \hat{\gamma}(k). \tag{35}$$

From this free energy we may calculate the quantum effective action. It is shown in fig. 5. In the infrared limit, $k \to 0$, it's boundary closes in and becomes infinitely confining. This is the proximity of the Gribov horizon in infrared directions.

CONCLUSION

We have understood quite a bit more about how Yang-Mills theory works in the minimal Landau gauge. We have resolved the apparent discrepancies from different approaches to determining the gluon propagator. We have found that they do not disagree, they just take limits in opposite order, and thus obtain consistently different results.

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