Maximal Abelian Gauge and Thermal Monopoles in SU(3) gauge theory

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Based on arXiv:1308.0302

1 – INTRODUCTION

Various mechanisms for color confinement are based on the condensation of topological objects (e.g. monopoles, vortices) in the QCD vacuum.
Such topological objects could be relevant also to explain many strongly interacting properties of the deconfined phase.

In the dual superconductor model (’t Hooft, 1975, Mandelstam, 1976), the condensation of Abelian magnetic monopoles is thought to induce dual Meissner effect.

The possible role played around and above T_c by thermal monopoles ”evaporating” from the zero T condensate has attracted a lot of attention in the last few years.
(Liao-Shuryak 2006; Chernodub-Zakharov, 2006; Ratti-Shuryak, 2009).
Thermal monopoles are identified with magnetic currents wrapping non-trivially around the thermal Euclidean time direction, which resemble the contributions to the path integral of a thermal particle ensemble. (M. Chernodub, V. Zakharov, 2006; V. Bornaykov, V. Mitrjushkin, M. Mueller-Preussker, 2002; S. Ejiri, 2006).

Extensive results obtained for thermal monopoles in the $SU(2)$ pure gauge theory:

(A. D'Alessandro, M. D. 2008; J. Liao, E. Shuryak 2008; M. Chernodub, A. D'Alessandro, M. D., V. Zakharov, 2009; A. D'Alessandro, M. D., E. Shuryak, 2010; V. Bornyakov, V. Braguta, 2011-2012; V. Bornyakov, A. Kononenko, 2012)

- Evidence for a magnetically dominated phase close to T_c and an electrically dominated phase at asymptotically high T
  - Magnetic charge increases logaritmically with T (electric-magnetic duality)
  - Thermal monopole density significant around T_c, but ρ/T^3 decreaeses logaritmically with T.
- The distribution of wrapping trajectories shows that thermal monopoles indeed condense at T_c, enforcing their link with color confinement.

The purpose of the present study is to extend such results to $SU(3)$.

- The maximal Abelian subgroup is $U(1) \times U(1)$: 2 different monopole species
- How to extend the Abelian projection in order to properly identify the two species?

Preliminary results on $SU(3)$ can be found also in V. G. Bornyakov, A. G. Kononenko and V. K. Mitrjushkin, PoS ConfinementX , 048 (2012).

2 – OUTLINE

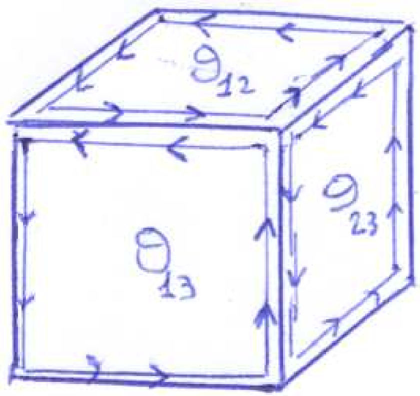
- **Abelian projection and Abelian monopoles on the lattice**

- **Maximal Abelian Gauge: extension from $SU(2)$ to $SU(N)$**

- **Numerical results for the $SU(3)$ pure gauge theory**

3 – Abelian monopoles on the lattice

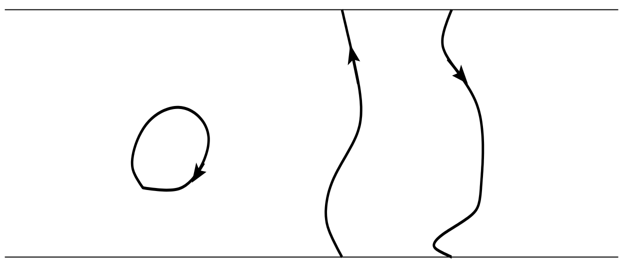
In compact $U(1)$ lattice gauge theory, magnetic monopoles are identified via the De Grand - Toussaint procedure. Let $u_\mu(n) \equiv e^{i\theta_\mu(n)}$ be the $U(1)$ link variables on a cubic 4D lattice, from which Abelian plaquettes are constructed $\theta_{\mu\nu} \equiv \hat\partial_\mu \theta_\nu - \hat\partial_\nu \theta_\mu$



Monopole currents are then constructed as

$$m_\mu = \frac{1}{2\pi} \varepsilon_{\mu\nu\rho\sigma} \hat\partial_\nu \overline{\theta}_{\rho\sigma} \;\; ; \qquad \theta_{\mu\nu} = \overline{\theta}_{\mu\nu} + 2\pi n_{\mu\nu}$$

i.e. one measures the net magnetic flux going out of a 3D cube, modulo Dirac string contributions



Monopole currents form closed loops, since $\hat\partial_\mu m_\mu = 0$. In a thermal theory, currents which wrap around the periodic time direction are identified with thermal monopoles. In a confining theory, wrappings take place also in spatial directions

4 – Abelian projection in $SU(N)$ gauge theories ('t Hooft, 1974, 1981)

From any normalized adjoint field $\phi(x) = \sum_a \phi^a(x) T^a$ one can define an Abelian ('t Hooft) tensor

$$F_{\mu\nu} = \mathrm{tr}\,(\phi\, G_{\mu\nu}) - (i/g)\mathrm{tr}\,(\phi\,[D_\mu \phi, D_\nu \phi])$$

In the gauge where $\phi(x)$ is constant and diagonal (unitary gauge), with $\phi(x) \propto \phi_0^k$

$$\phi_0^k = \frac{1}{N}\,\mathrm{diag}\,(\underbrace{N-k, \ldots, N-k}_{k}, \underbrace{-k, \ldots, -k}_{N-k}) \quad \text{(fundamental weights)}$$

$F_{\mu\nu}$ becomes a standard e.m. tensor (Del Debbio, Di Giacomo, Lucini, Paffuti, 2002)

$$F_{\mu\nu}^{(k)} = \mathrm{tr}\,\left(\partial_\mu(\phi_0^k A_\nu) - \partial_\nu(\phi_0^k A_\mu)\right) \equiv \partial_\mu a_\nu^{(k)} - \partial_\nu a_\mu^{(k)} \;; \quad a_\mu^{(k)} \equiv \mathrm{tr}(\phi_0^k A_\mu) = \sum_{j=1}^{k} (A_\mu)_{jj}\,,$$

If A_μ^D is the diagonal part of the gauge field, then

$$A_\mu^D = \sum_{k=1}^{N-1} a_\mu^{(k)} \alpha^k \;; \quad \alpha^k = \frac{1}{2}\mathrm{diag}\,(0, 0, \ldots, 0, \overbrace{1, -1}^{k,k+1}, 0, \ldots, 0) \;\; (\mathrm{k-th\,simple\,root})$$

$F_{\mu\nu}^{(k)}$ is the e.m. tensor for the $U(1)$ subgroup generated, in the unitary gauge, by α^k.

In $SU(N)$, one can identify $N-1$ independent Abelian subgroups ($U(1)^{(N-1)}$):

- **Choose an hermitean, traceless adjoint Higgs field $X(x)$: $X(x) \to G(x)X(x)G^{-1}(x)$**

- **Fix the gauge where $X(x) = X^D(x) = \mathrm{diag}(X_1(x), X_2(x), \ldots, X_N(x))$ with $X_j(x) \geq X_{j+1}(x)$. That fixes the gauge apart from a residual $U(1)^{(N-1)}$ gauge symmetry.**

- **Associate a 't Hooft tensor $F^{(k)}_{\mu\nu}$ to each residual $U(1)$ group: $\phi(x) = \phi_0^k$ in the diagonal gauge.**
  All tensors are mutually neutral (the $N-1$ subgroups are independent)

- **In general, one can write**

$$X^D(x) = \sum_{k=1}^{N-1} c^k(x)\phi_0^k \; ; \qquad c^k(x) = \frac{1}{2}\mathrm{tr}(\alpha^k X^D(x)) = X_k(x) - X_{k+1}(x)$$

Points x_0 where $c_k(x_0)$ vanishes ($X_k(x_0) = X_{k+1}(x_0)$) define the location of a magnetic monopole in the k-th $U(1)$ subgroup:

- **The residual $U(1)$ is enlarged to a full $SU(2)$ subgroup (α^k being the corresponding $\sigma^3/2$) in x_0.**

- **around x_0, one can either choose an hedgehog solution for $X(x)$ ('t Hooft, Polyakov, '74) or, in the unitary gauge, a solution where $X(x)$ is diagonal and the field $a_\mu^{(k)}$ contains the contribution from a Coulomb-like magnetic field centered around x_0.**

- **In a lattice setup, looking for points where two eigenvalues coincide is ill defined. One then works in the diagonal gauge and looks for monopole fields via the De Grand - Toussaint procedure.**

5 – Maximal Abelian Gauge (MAG) Projection in $SU(2)$ and extension to $SU(N)$

For $SU(2)$, MAG is the gauge where the following functional has a maximum

$$F_{\rm MAG} = \sum_{\mu,n} \textbf{tr} \left(U_\mu(n) \sigma_3 U_\mu^\dagger(n)\, \sigma_3 \right) = \sum_{\mu,n} 2 \left(|U_\mu(n)_{11}|^2 + |U_\mu(n)_{22}|^2 - 1 \right)$$

On stationary points of $F_{\rm MAG}$, the diagonal Hermitean, traceless Higgs field is

$$X^{\rm MAG}(n) = \sum_\mu \left[U_\mu(n) \sigma_3 U_\mu^\dagger(n) + U_\mu^\dagger(n-\mu) \sigma_3 U_\mu(n-\mu) \right] \ ,$$

The explicit form of the Higgs field $X^{\rm MAG}$ is known only after MAG has been fixed and is, in fact, Gribov copy dependent.

Part of the popuparity of the MAG projection is due to the fact that abelian projected fields retain most of the original dynamics (Abelian Dominance).

The properties of magnetic monopoles defined after MAG projection also show a nice scaling to the continuum limit.

Extension to $SU(3)$

A standard extension adopted for $SU(3)$ is (A. S. Kronfeld, G. Schierholz and U. J. Wiese, 1987)

$$
\begin{aligned}
F_{\mathrm{MAG}}^{SU(3)} &= \sum_{\mu,n}\left(\textbf{tr}\left(U_\mu(n)\lambda_3 U_\mu^\dagger(n)\,\lambda_3\right) + \textbf{tr}\left(U_\mu(n)\lambda_8 U_\mu^\dagger(n)\,\lambda_8\right)\right) \\
&= 2\sum_{\mu,n}\left(|(U_\mu(n))_{11}|^2 + |(U_\mu(n))_{22}|^2 + |(U_\mu(n))_{33}|^2 - 1\right)
\end{aligned}
$$

This is the natural extension of Maximal Abelian Gauge for issues regarding Abelian dominance, but has some problems for Abelian projection:

- **No diagonal Higgs field is naturally associated to it**
- **On extremal points, the residual symmetry is not just $U(1)^{(N-1)}$, since global permutations of group indexes leave $F_{\mathrm{MAG}}^{SU(3)}$ invariant.**
  Such permutations mix the two Abelian charges, which become ill-defined.

Possible alternatives? Yes, generalized MAG (J. D. Stack, W. W. Tucker 2002)

Generalized MAG for $SU(N)$ gauge theories

$$\tilde{F}_{\rm MAG} = \sum_{\mu,n} \textbf{tr} \left(U_\mu(n)\tilde{\lambda}U_\mu^\dagger(n)\,\tilde{\lambda} \right) \quad ; \quad \tilde{\lambda} = {\rm diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots \tilde{\lambda}_N)\,,$$

where $\tilde{\lambda}$ is a generic element of the Cartan subalgebra. Some properties (see arXiv:1308.0302):

- **A diagonal Higgs field exists, provided $\tilde{\lambda}$ has no pair of coinciding eigenvalues**

$$\tilde{X}(n) = \sum_\mu \left[U_\mu(n)\tilde{\lambda}U_\mu^\dagger(n) + U_\mu^\dagger(n-\mu)\tilde{\lambda}U_\mu(n-\mu) \right]\,.$$

- **If we require $X_j(n) \geq X_{j+1}(n)$ around the perturbative vacuum, then the condition reads**

$$\tilde{\lambda} = \sum_{k=1}^{N-1} b^k \phi_0^k \;; \qquad b^k > 0 \;\; \forall\, k$$

- **Residual symmetry is strictly $U(1)^{(N-1)}$ (no symmetry under permutations)**
- **No coinciding eigenvalues of $X(n)$ around the perturbative vacuum: $X(n) = \sum_{k=1}^{N-1} c^k(n)\phi_0^k$ with $c^k \sim b^k$. $\Longrightarrow$ the appearance of monopoles requires non-perturbative fluctuations**

We adopt $b^k = 1\ \forall\, k$, which treats all monopole species symmetrically:

$$\tilde{\lambda} = \sum_{k=1}^{N-1} \phi_0^k = {\rm diag}\left(\frac{N-1}{2}, \frac{N-1}{2} - 1, \frac{N-1}{2} - 2, \ldots, -\frac{N-1}{2} \right)\,.$$

Summary for $SU(3)$

- **Gauge is fixed by maximization of the functional**

$$\tilde{F}_{\mathrm{MAG}} = \sum_{\mu,n} \mathbf{tr}\left(U_\mu(n)\tilde{\lambda}U_\mu^\dagger(n)\,\tilde{\lambda}\right) \;\; ; \quad \tilde{\lambda} = \frac{1}{3}\mathrm{diag}(1,\ 0,\ -1)\,,$$

a standard local over-relaxed algorithm is adopted, working over $SU(2)$ subgroups.

- **On the gauge fixed configuration, the diagonal of gauge links is extracted,**

$$U_\mu^D(n) = \mathrm{diag}(e^{i\phi_\mu^1(n)}, e^{i\phi_\mu^2(n)}, e^{i\phi_\mu^3(n)})$$

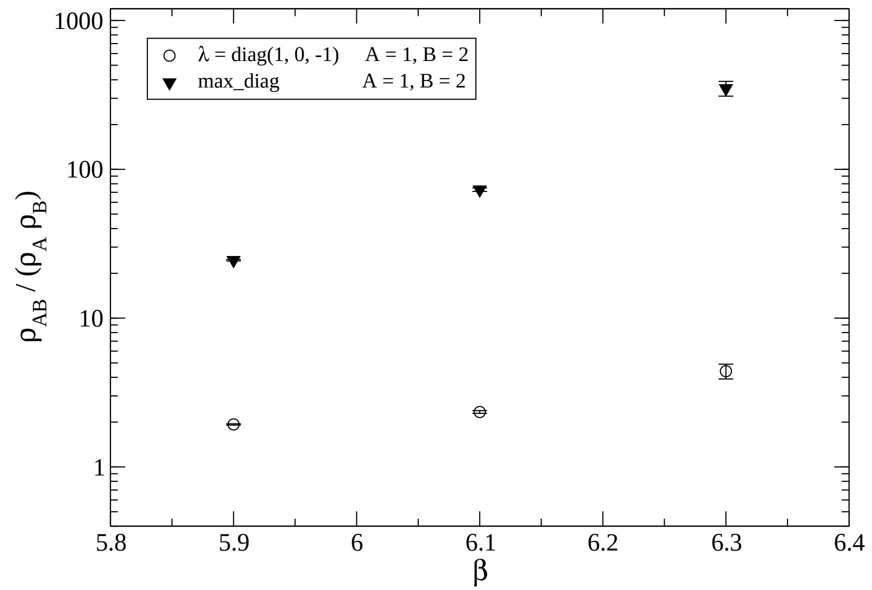
where $U_\mu^D(n)$ is the diagonal $SU(3)$ matrix maximizing $\mathrm{Re}(\mathrm{tr}(U_\mu^D(n)U_\mu^\dagger(n)))$

- **The two Abelian phases are then extracted according to**

$$\theta_\mu^1(n) = \phi_\mu^1(n) \qquad \theta_\mu^2(n) = \phi_\mu^1(n) + \phi_\mu^2(n) = -\phi_\mu^3(n)$$

the monopole currents m_μ^1 and m_μ^2 are then determined following the De Grand-Toussaint method.

Are the two monopole species really independent?



Probability of two coinciding monopole currents, normalized by the product of the corresponding single current probabilities for different values of β (16^4 lattice) and for different choices of the monopoles currents:
should be of $O(1)$ for independent monopoles species (monopole interactions can account for moderate deviations from 1)

This is the case for our definition, but it is not the case for the standard extension of the MAG which maximizes the diagonal part of gauge links

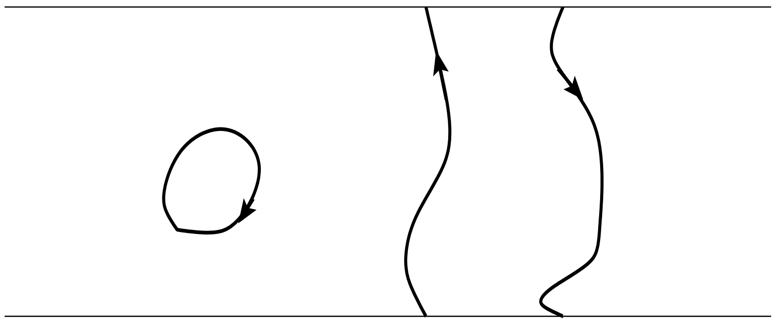
6 – Numerical Results

- **We have discretized the $SU(3)$ pure gauge theory in the Wilson (plaquette) formulation, exploring various lattice spacings and lattice sizes, ranging from $24^3 \times L_t$ to $48^3 \times L_t$, with $4 < L_t < 11$.**

- **The lattice spacing has been fixed by tuning the inverse gauge coupling $\beta = 2N/g_0^2$ according to the non-pertubative β-function determined in determined according to G. Boyd *et al*, hep-lat/9602007**

- **Most properties that I am going to show resemble very closely, for each monopole species separately, those of $SU(2)$ thermal monopoles.**

- **New features however appear, related to the interactions among the two different monopole species.**

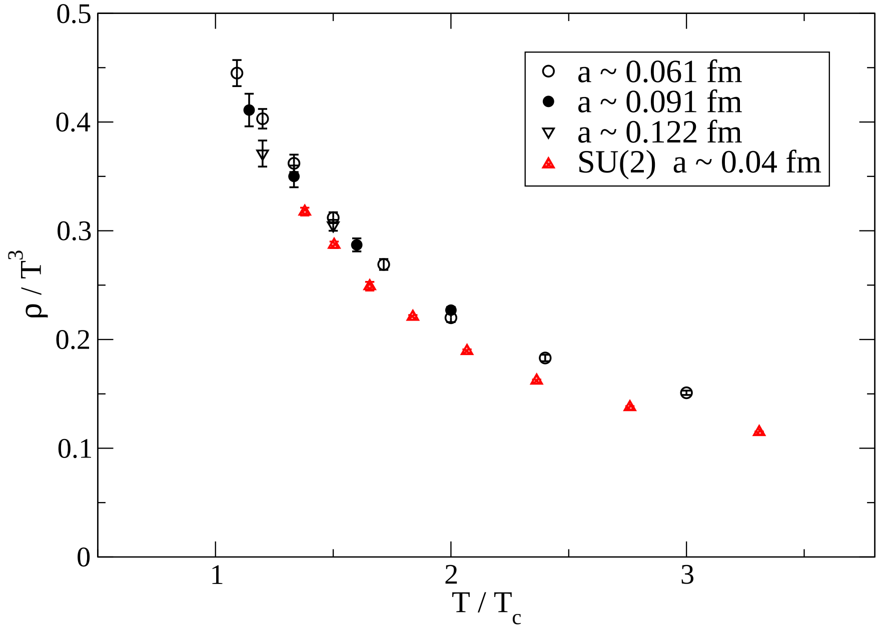


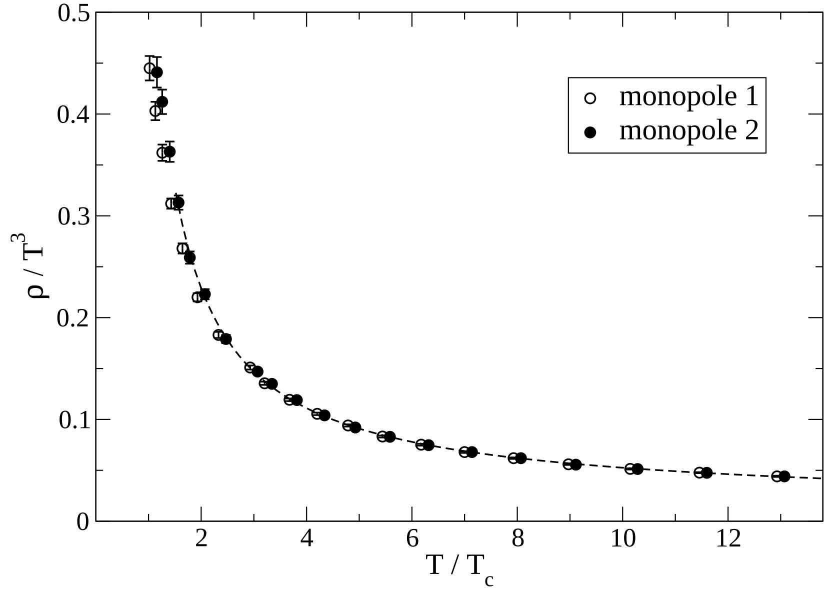
For each configuration, we locate monopole currents with non-trivial winding number in time, and their position at a given reference time slice. After that, we can investigate various quantities.

Density of thermal monopoles

$$\rho = \frac{\langle \sum_{\vec{n}} |N_{wrap}(m_0(\vec{n},t))| \rangle}{V_s}$$

$N_{wrap}(m_0(\vec{x},t))$ **is the winding number of the current** $m_0(\vec{x},t)$**,** $V_s = (L_s a)^3$ **is the spatial volume**



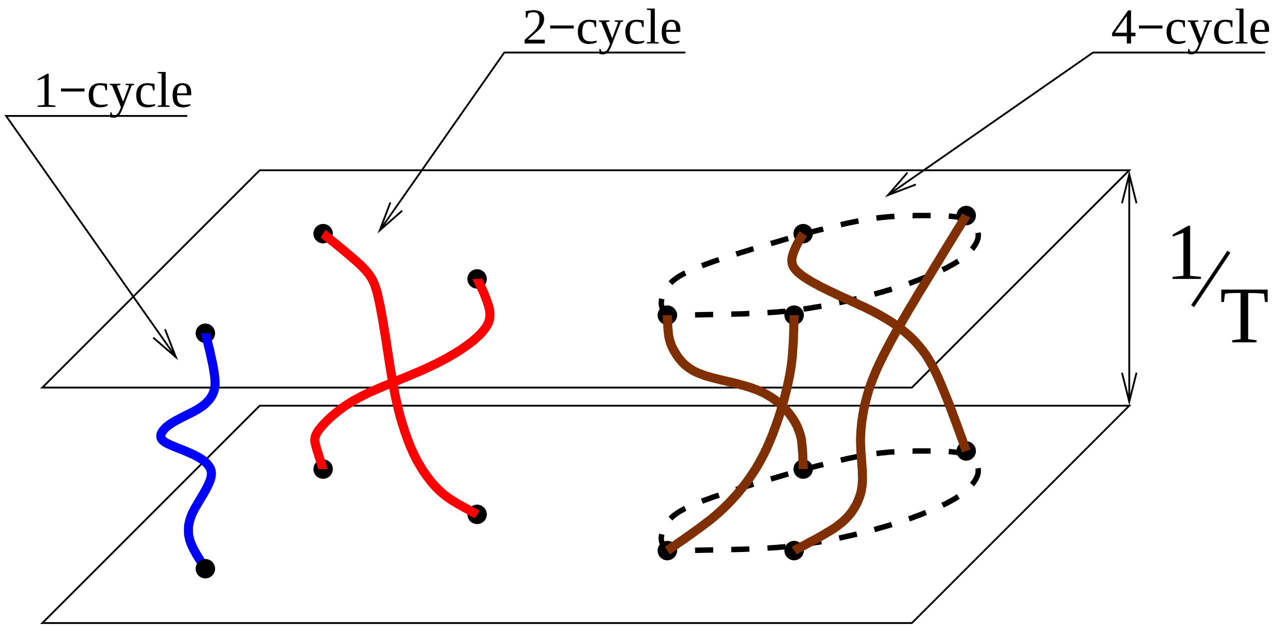


- **Monopole densities of the two species coincide within errors. Nice scaling to the continuum limit**

- **Results very close to those of $SU(2)$ pure gauge theory**

- **Asymptotically, ρ/T^3 decreases $\propto (1/\log T)^3$, meaning that monopoles become irrelevant in the perturbative regime, while they play a significant role around the transition $\rho/T^3 \sim O(1)$.**

Distribution of trajectories with multiple wrappings

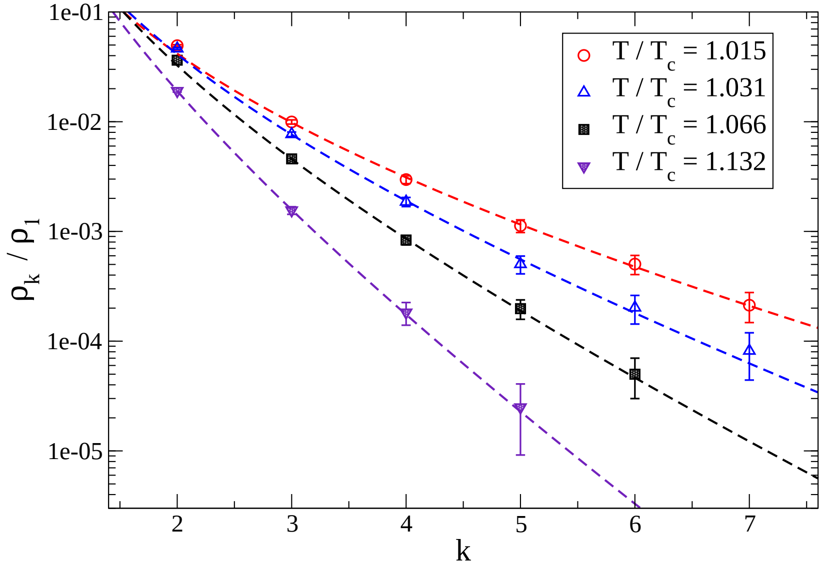


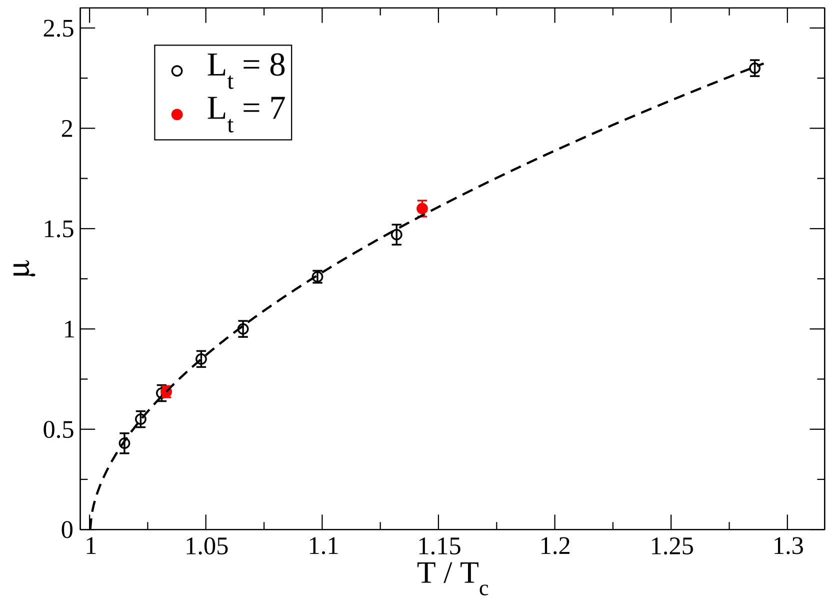
Like for the path integral of a system of identical particles, monopole trajectories with multiple windings in the time direction can be associated to two-(or multiple)-particle exchange.

Their distribution as a function of T can be used to investigate thermal monopole condensation.

(M. Cristoforetti and E. Shuryak, arXiv:0906.2019)

(A. D'Alessandro, M.D., E. Shuryak, arXiv:1002.4161)





- **The density of trajectories winding k-times, ρ_k, is negligible, for $k > 1$, at high T. It becomes significant only as one approaches T_c from above.**

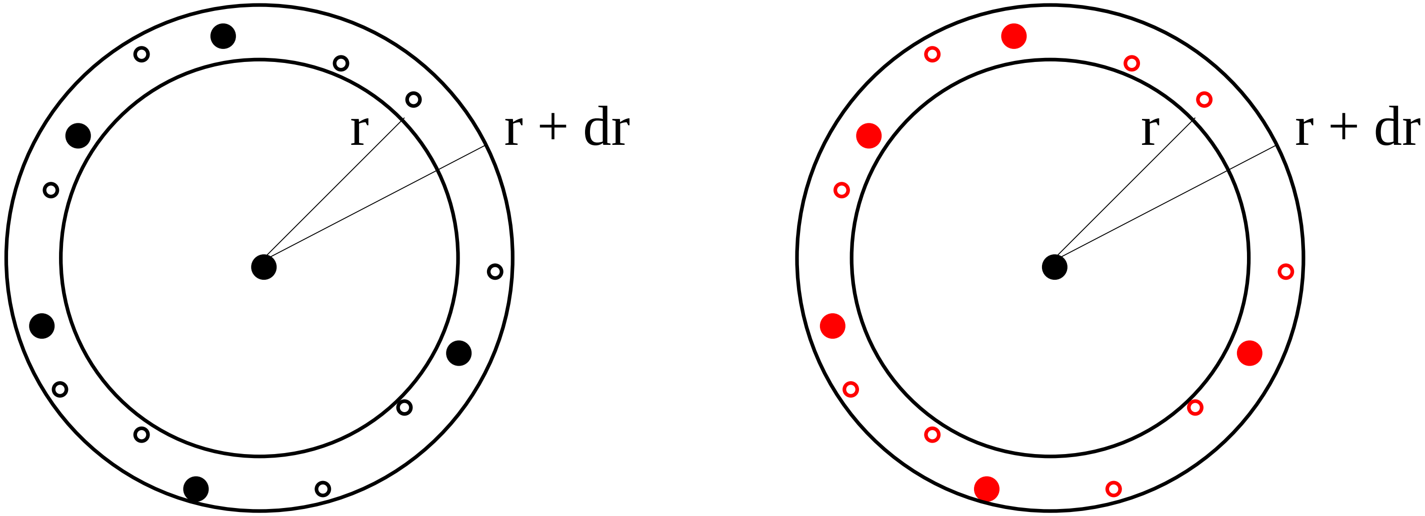
- **For a system of free bosons, if $\mu = -T\hat{\mu}$ is the chemical potential,**

$$\rho_k \propto e^{-\hat{\mu} k}/(\lambda^3\, k^{5/2})$$

$\mu \longrightarrow 0$ **signals Bose-Einstein condensation (BEC)**

- **If we assume such simple ansatz for the monopole ensemble, we can extract $\hat{\mu}$ and obtain that, like for $SU(2)$, BEC takes place at T_c (within errors) for both species. A fit $\hat{\mu} = A\,(T - T_{\rm BEC})^{\nu'}$ gives $T_{\rm BEC} = 1.0003(36)\, T_c$**

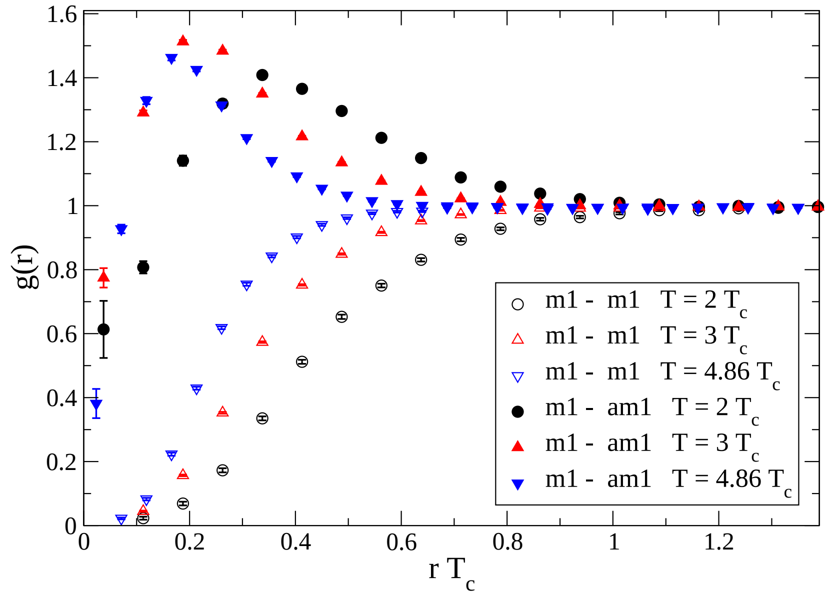
Monopole interactions

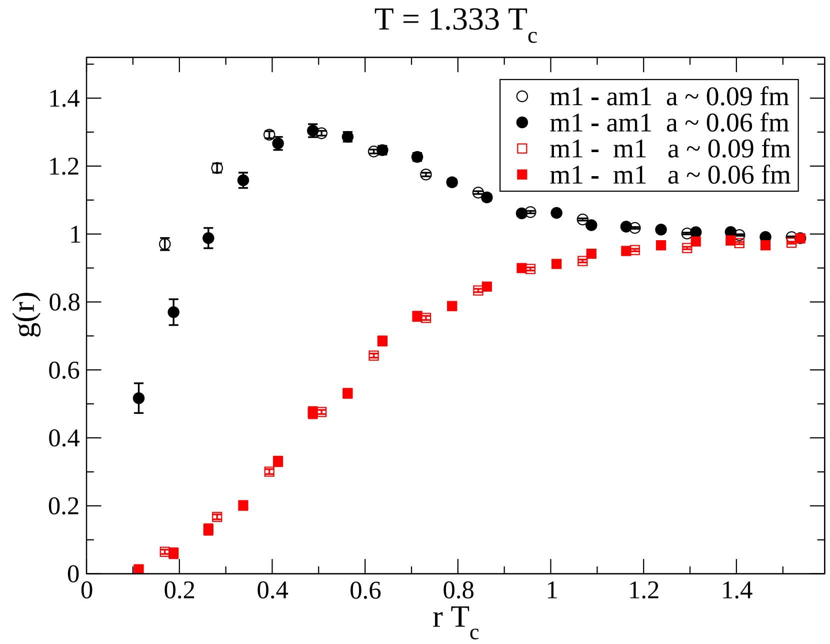


$\Longrightarrow$ **fix a reference monopole, count monopoles (antimonopoles) of the same (or different) species at distance $\in [r, r + dr]$ and normalize by the same number expected for a random (uncorrelated) distribution:**

that gives the correlation function $g(r)$

- $g(r) = 1 \Longrightarrow$ **no interaction**
- $g(r) \neq 1 \Longrightarrow$ **non-trivial interaction**





- **monopole-(anti)monopole of the same species repel (attract) each other.**
- **the large distance behavior of $g(r)$ gives information about the interaction potential,**

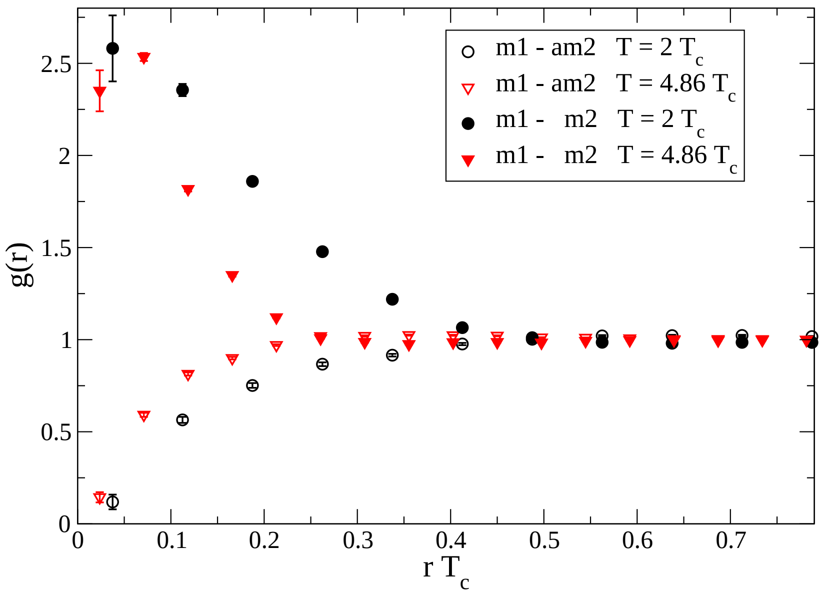
$$g_{AB}(r) \simeq \exp(-V_{AB}(r)/T) \, .$$

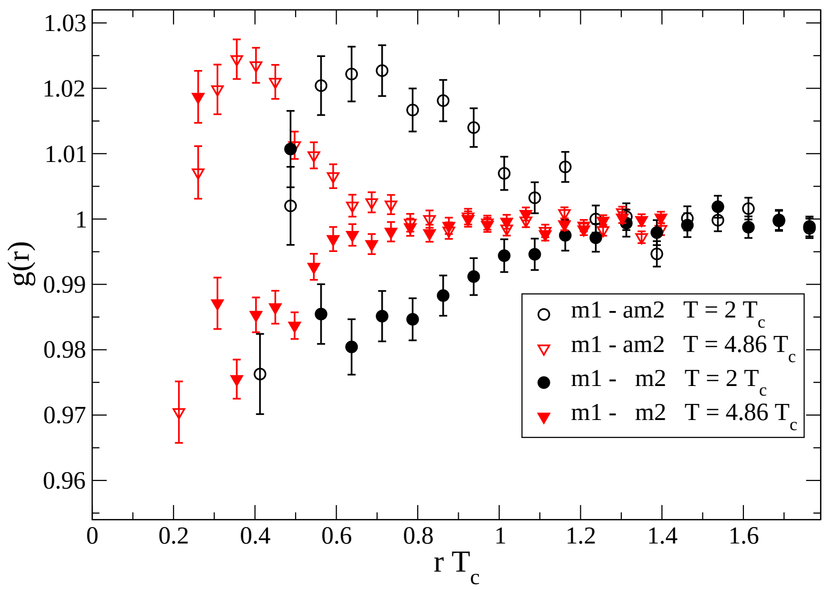
As for $SU(2)$, a Coulomb screened potential describes well numerical data

$$V_{AB}(r) = \pm \frac{\alpha_M e^{-\lambda_P r}}{r} \, ,$$

α_M **increases (**λ_P **decreases) with increasing** T**.**

Correlations among different species reveal new features





- **at short distances, interaction sign reversed: monopole 1 attracts monopole 2. OK, interaction between monopole k and monopole k' is $\propto$Tr $(\alpha^k \alpha^{k'})$.**
- **at intermediate distances, $g(r) - 1$ changes sign! Oscillating behavior of $g(r)$ indicates non-trivial structures.**
- **monopole1-monopole2 molecules? Are we seeing the monopole constituents of $SU(N)$ calorons?**

7 – Conclusions and Perspectives

- We have discussed an extension of MAG projection to $SU(N)$, with the aim of detecting $(N-1)$ independent monopoles
- Results on thermal monopoles known from $SU(2)$ are confirmed for $SU(3)$.
- New features appear for $SU(3)$, related to the interactions among different monopole species. Apart from an obvious extension to full QCD, that claims for an extension to $SU(N)$ with $N > 3$. Open questions:
  - for $N > 3$ the deconfinement transition is strong first order, then one would expect $T_{BEC} \neq T_c$, in particular $T_{BEC} < T_c$.
  - Do the $(N-1)$ different monopoles form bound states? Can those be related to calorons?
  - Is it possible, along this way, to relate thermal monopole condensation to the drastic change of θ dependence taking place at T_c? (C. Bonati, M. D., H. Panagopoulos and E. Vicari, arXiv:1301.7640)
- Finally, it would be interesting to extend the analysis for $SU(N)$ to Abelian projections based on different gauges, e.g., the Laplacian gauge (P. de Forcrand, M. Pepe, 2001)