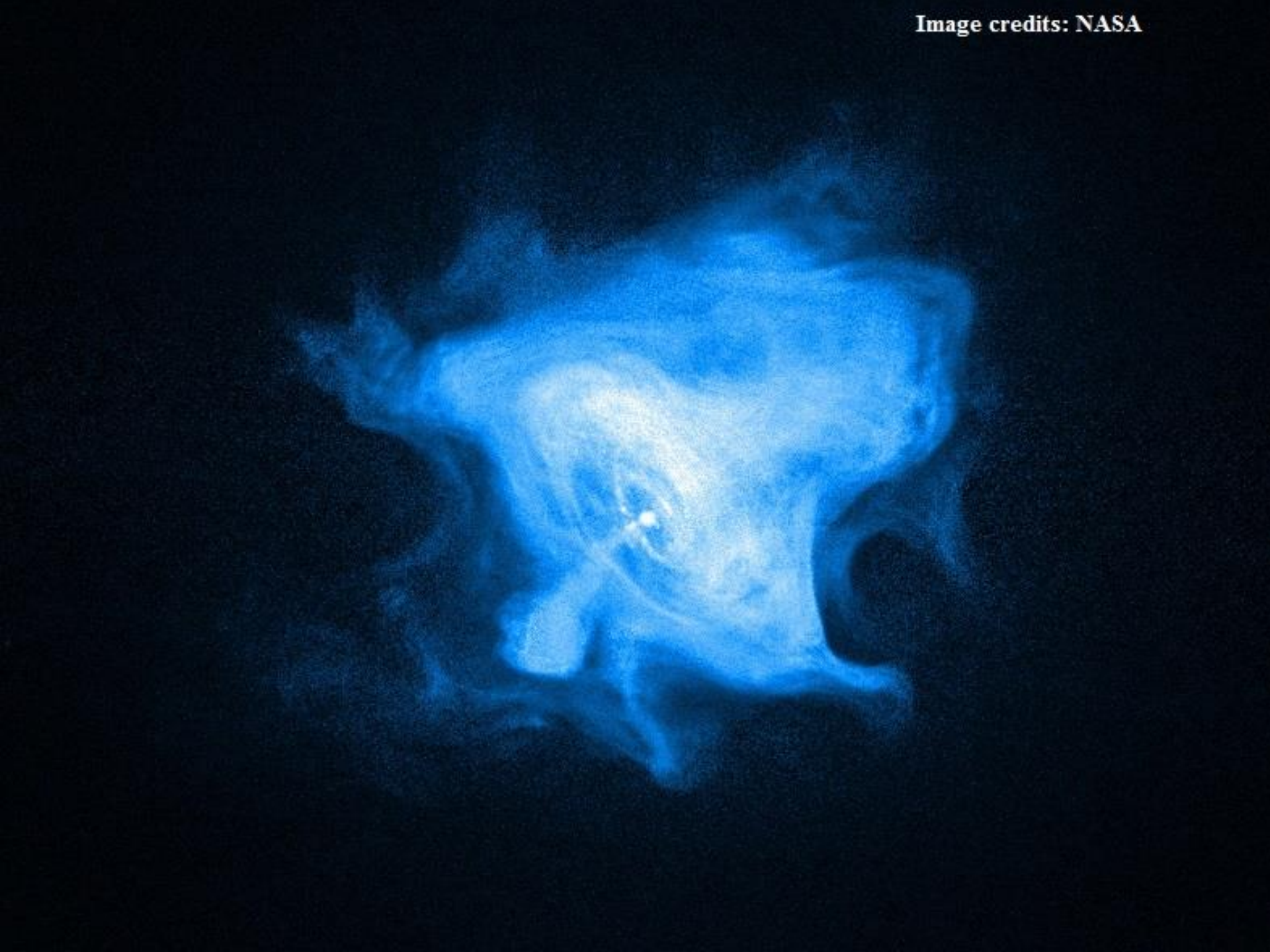
Image credits: NASA

Instability of the Lattice in the Inner Crust of Neutron Stars

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ECT* Workshop

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Based on

D. Kobyakov & C.J. Pethick

Phys. Rev. C **87**, 055803 (13 May 2013).

"Dynamics of the inner crust of neutron stars: Hydrodynamics, elasticity, and collective modes"

arXiv:1309.1891 [nucl-th] (6 September 2013).

"Towards a Metallurgy of Neutron Stars"

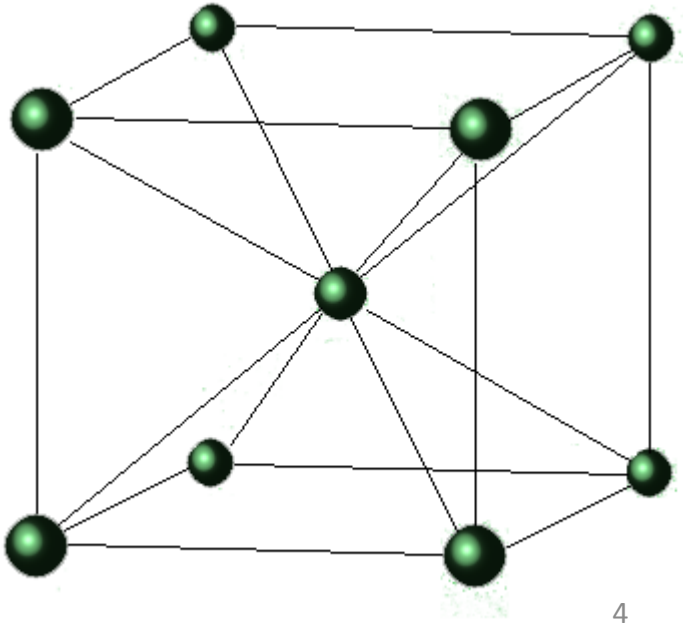
Outline

Approach

Calculate collective mode frequencies at long and short wavelengths. *Important*: neutron-proton interactions, electron screening of ions, and anisotropy of the microscopic crystal structure.

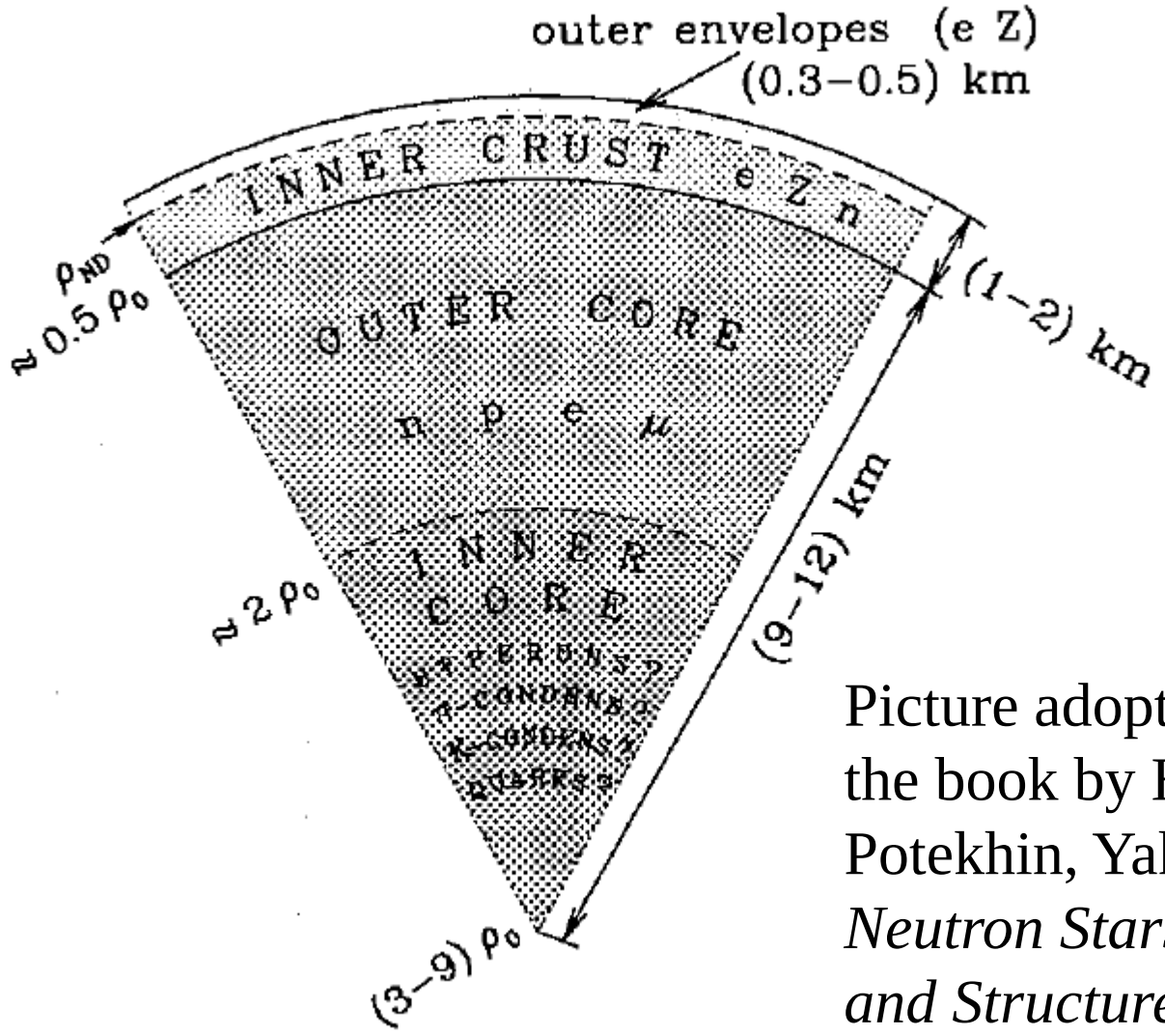
Main result

The bcc structure, considered traditionally as the equilibrium configuration, is **unstable**.



Part I. Collective modes of the inner crust

Standard Picture of NS Structure



Picture adopted from the book by Haensel, Potekhin, Yakovlev, *Neutron Stars I: EOS and Structure* (2007)

Essential ingredients of the Inner Crust

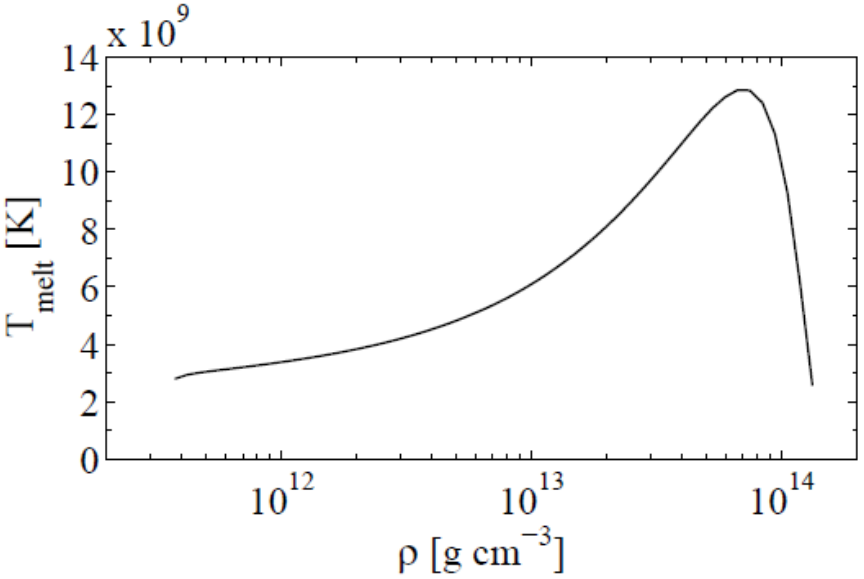
- (1) Ultrarelativistic almost free electron gas
- (2) Neutron rich nuclei
- (3) Neutron fluid outside nuclei – highest neutron energy levels not bound to nuclei.

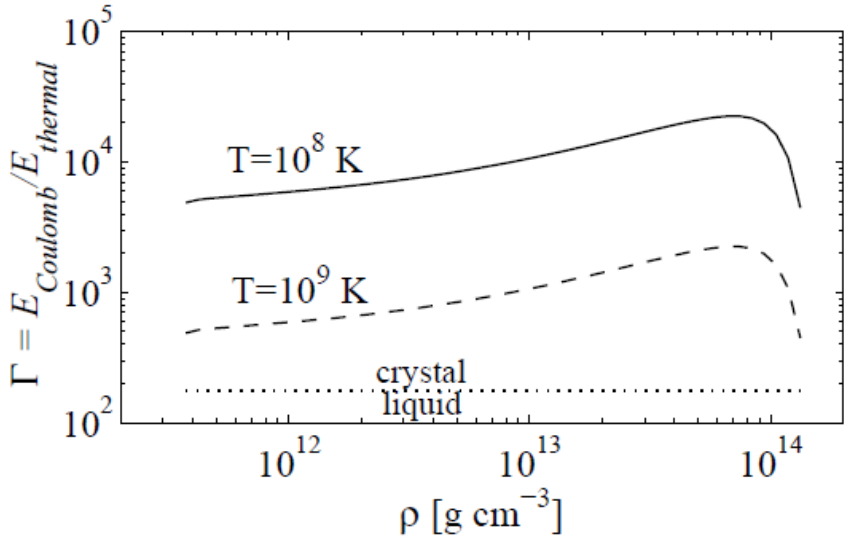
(1) Ultrarelativistic electron gas

- Charge-neutral matter
- Beta-equilibrium (not a necessary requirement)
- Uniform electron density (for the system ions+electrons)
- Electron screening wavenumber is smaller than the Debye wavenumber of the ionic lattice
- Electron plasma frequency is larger than frequencies of the ionic lattice

(2) Neutron-rich nuclei

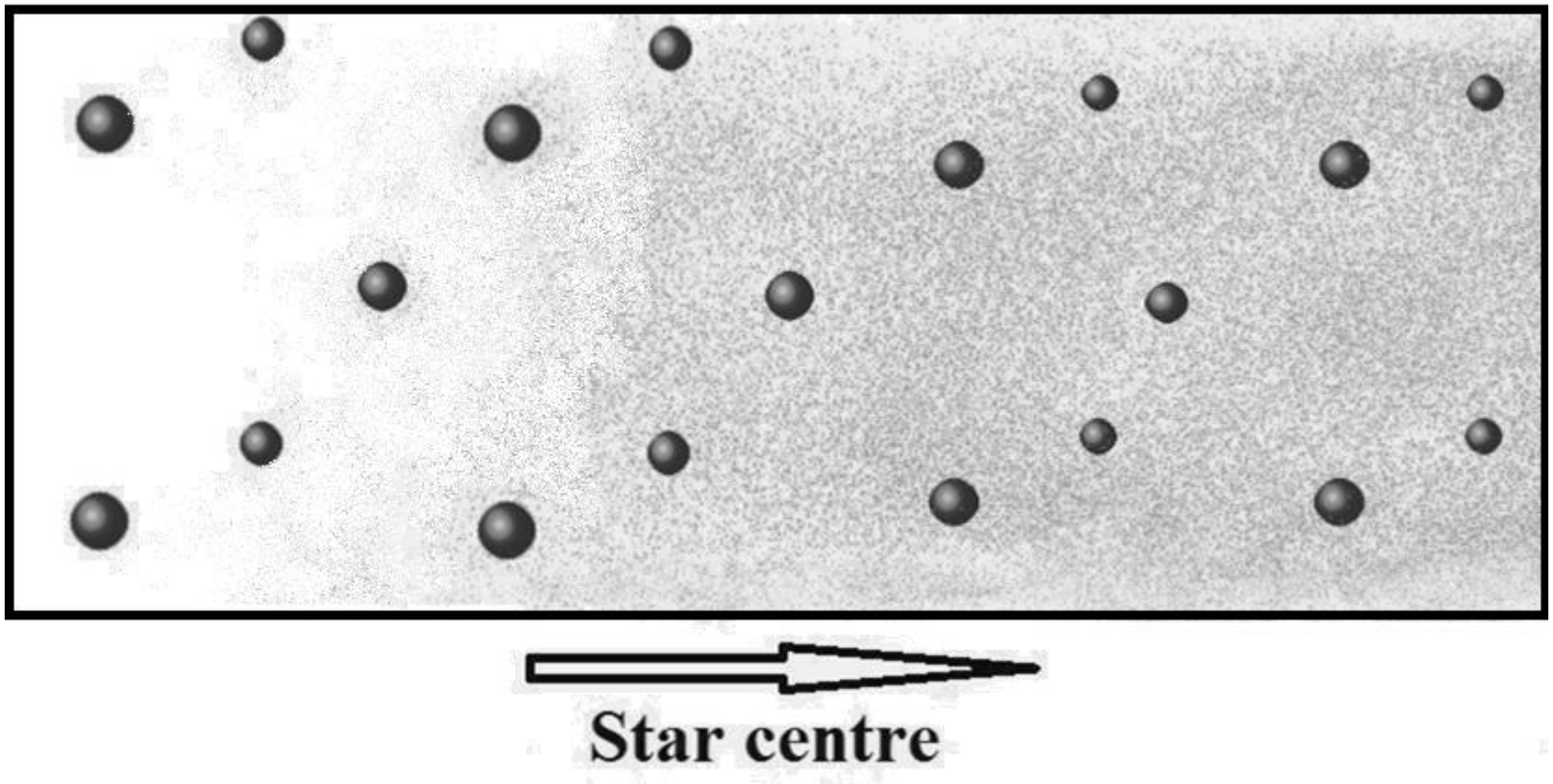
- Typical nuclei: $Z \approx 40$, $A \approx 150$
- Stabilized against beta-decay by neutron fluid
- Temperature of inner crust less than T_{melt}: Coulomb crystal





(3) Neutron fluid

- Degenerate Fermi system
- Superfluidity (theoretical & observational evidence)



Why bcc is expected to be the ground state?

- The Coulomb energy is minimized for bcc configuration for point nuclei and for a uniform electron background

  (Also agrees with what one finds for the Coulomb crystal structure when the crust disappears at the inner boundary of the inner crust [G. Baym, H. A. Bethe, C. J. Pethick, *Nucl. Phys. A* **175**, 225 (1971)])

Hydroelasticity of the inner crust

- Continuity equations (linearized)

$$\partial_t \delta n_p + n_p \boldsymbol{\nabla} \boldsymbol{v}_p = 0,$$

$$\partial_t \delta n_n + n_n^s \boldsymbol{\nabla} \boldsymbol{v}_n + n_n^n \boldsymbol{\nabla} \boldsymbol{v}_p = 0.$$

- Euler equations (assuming that solid is isotropic)

$$m \partial_t \boldsymbol{v}_n + \boldsymbol{\nabla}\left(E_{nn}\delta n_n + E_{np}\delta n_p\right) = 0,$$

$$m\left(n_p + n_n^n\right)\partial_t \boldsymbol{v}_p + n_n^n \boldsymbol{\nabla}\mu_n + n_p \boldsymbol{\nabla}\left(\mu_p + \mu_e\right) -$$

Isotropic (pressure)

$$-\frac{4}{3} S\, \boldsymbol{\nabla}(\operatorname{div} \boldsymbol{u}) + S\, \boldsymbol{\nabla} \times (\operatorname{rot} \boldsymbol{u}) = 0,$$

Shear

$$\operatorname{div} \boldsymbol{u} = -\frac{\delta n_p}{n_p}.$$

Evaluation of parameters

- Thermodynamic derivatives E_{ij}, and density of the dripped neutrons – from the compressible liquid drop model of Lattimer and Swesty.

- Shear modulus – from calculations of the Coulomb crystal properties [Ogata, Ichimaru (1990)].

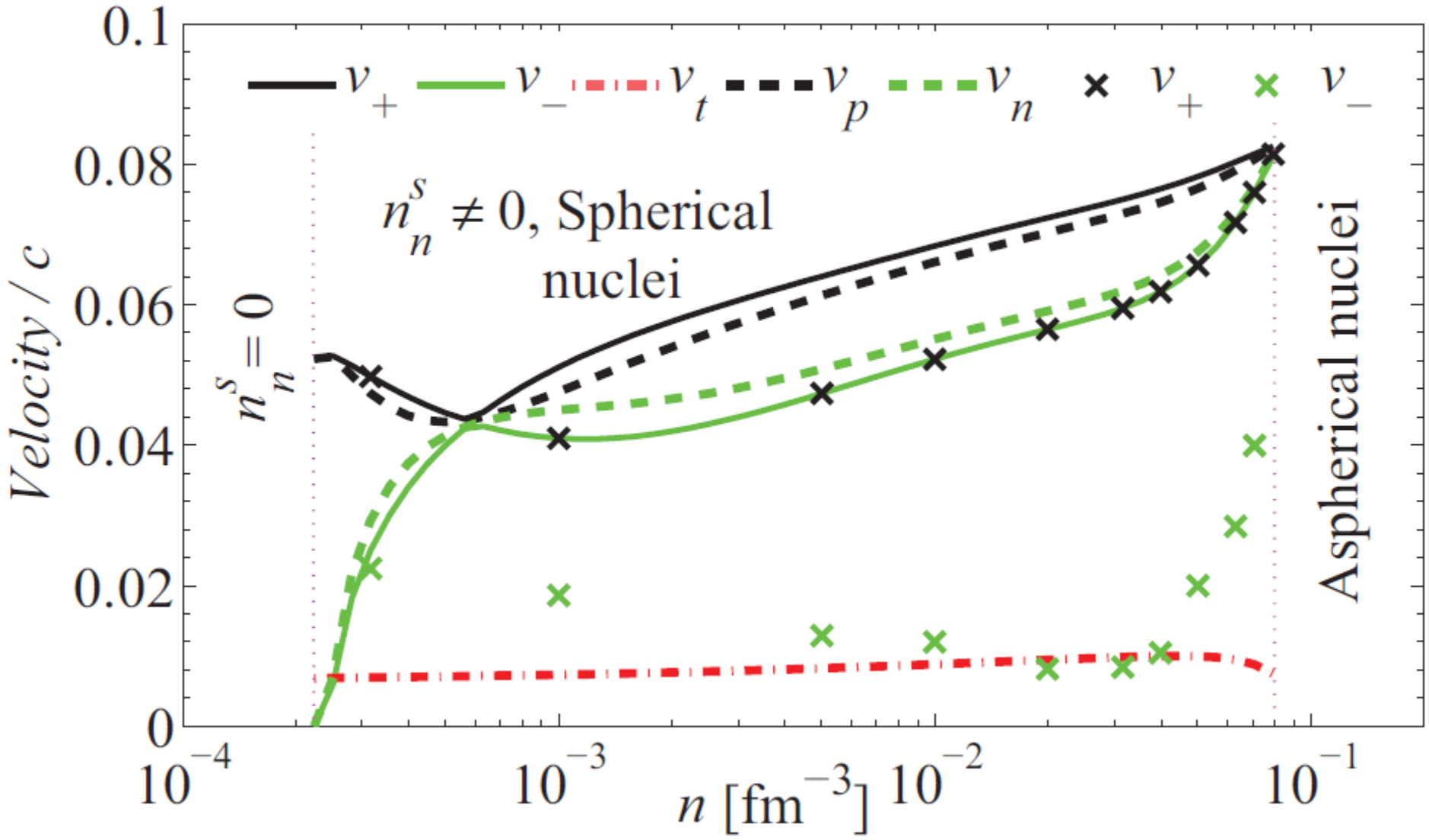
Modes

- Velocities, density deviations, and deformation $\sim e^{i(\omega t - \boldsymbol{q}\boldsymbol{r})}$
- Transverse modes $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$: $v_t^2 = \dfrac{S}{m\left(n_p + n_n^n\right)}$.
- Longitudinal modes $\boldsymbol{\nabla} \times \boldsymbol{u} = 0$:

$$v_p = 0 \text{ or } v_n^s = 0 \implies v_n \text{ and } v_p \text{ (uncoupled modes)}$$

$$v_+ \text{ and } v_- \text{ (coupled modes)}$$

Collective mode velocities



Part II. Microscopic structure of the inner crust

Stability of matter: Long wavelengths

- Stability condition: $\delta E = \sum_{i,j=n,p+e} \frac{\partial \mu_j}{\delta n_j} \delta n_i \delta n_j > 0$.

Induced p-p interaction

- Equivalently: $E_{nn} > 0$, $E_{pp} - \frac{E_{np}^2}{E_{nn}} = \left. \frac{\partial \mu_p}{\partial n_p} \right|_{\mu_n} > 0$.

Direct p-p interaction

- Long-wavelength perturbations are stable, since electrons provide a large positive contribution to the effective proton-proton interaction.

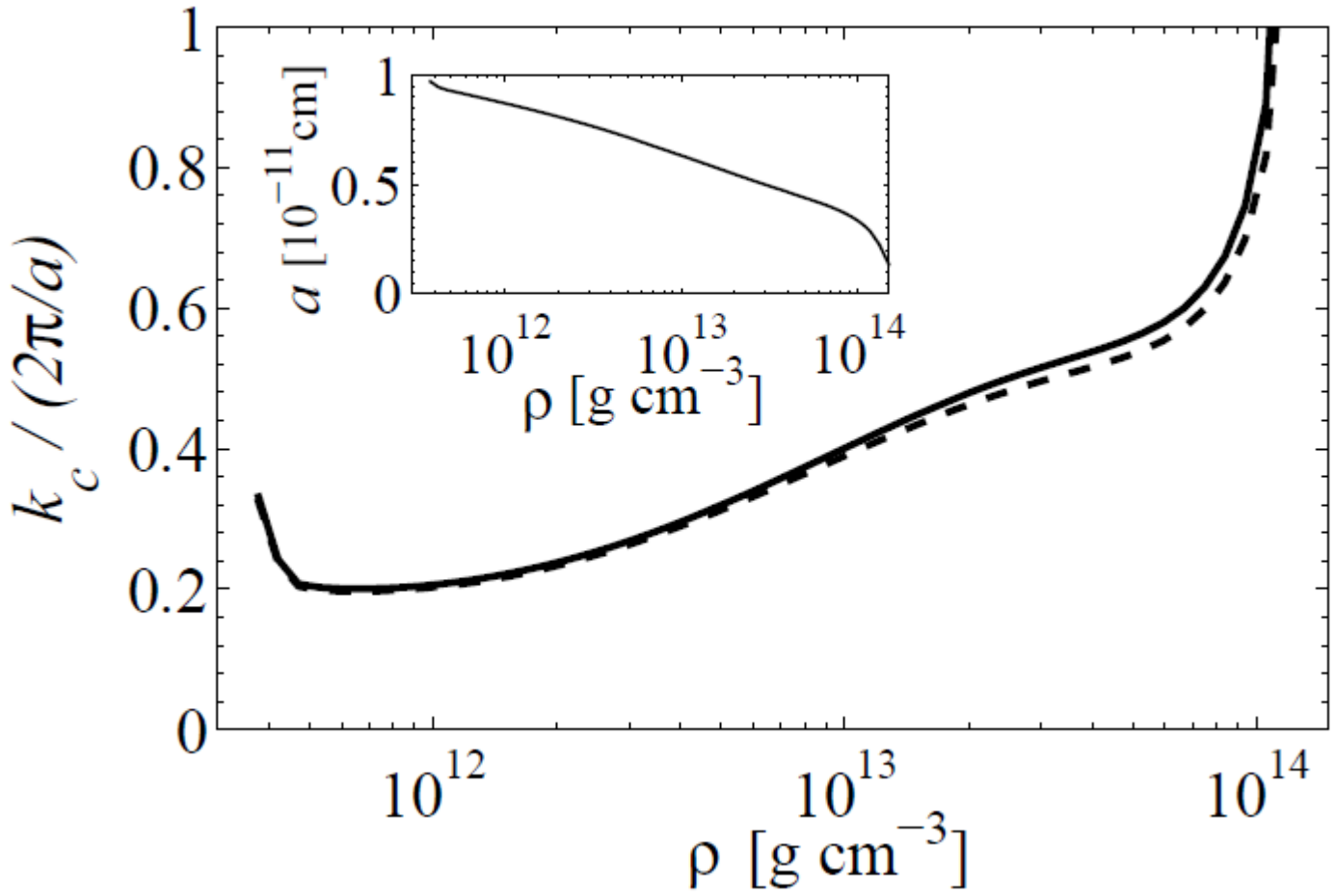
Stability of matter: Short wavelengths

- Compressional short wavelength perturbations of lattice density are not screened by electrons.
- Effective proton-proton interaction is modified by the screening corrections:

$$E_{pp}(k) = E_{pp} - \frac{\partial \mu_e}{\partial n_e} \frac{k^2}{k_{\mathrm{FT}}^2 + k^2}$$

- Other quantities (E_{nn}, E_{np}) are assumed to be independent on the wavenumber.

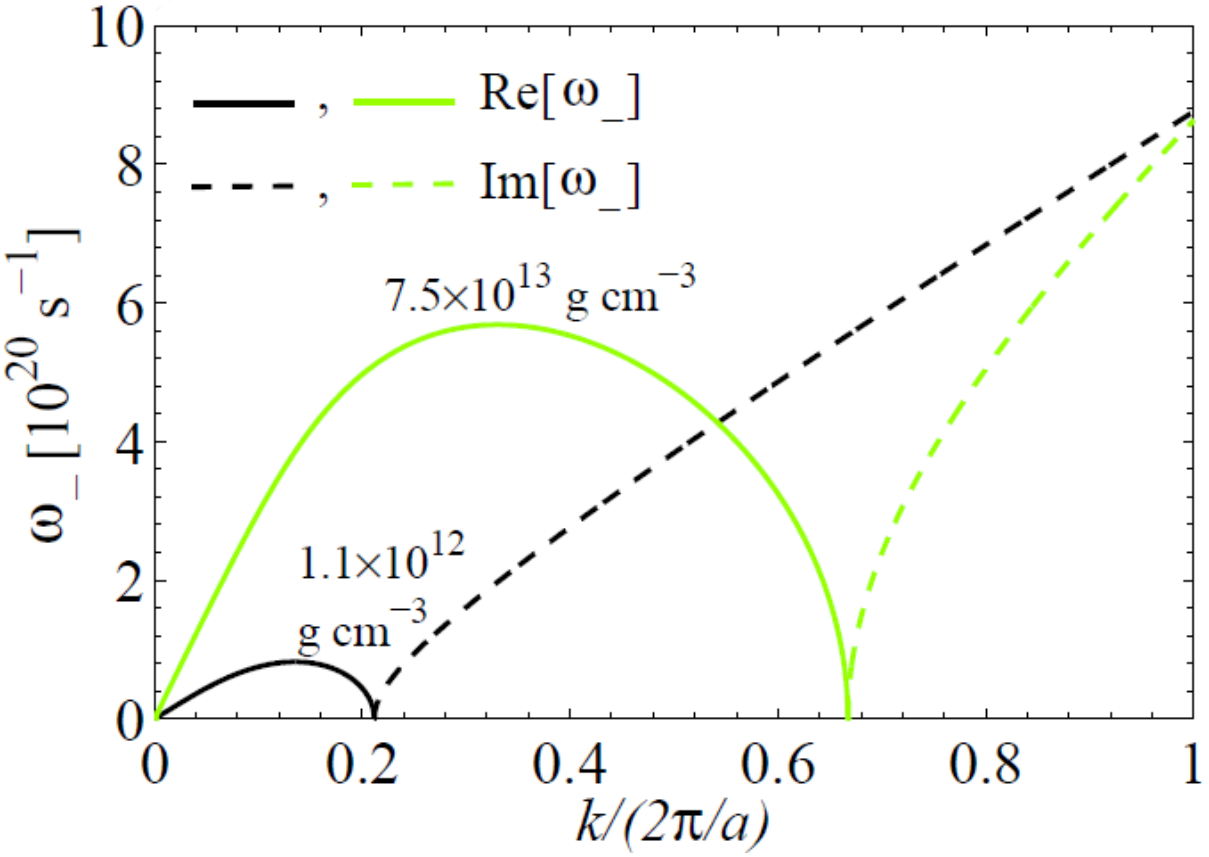
Critical wavenumber



Which wavenumber is the most unstable?

- As the lowest approximation we us $E_{pp}(k)$ in the long-wavelength isotropic model of the collective modes.
- The in-phase mode becomes unstable.

Dispersion relation (with screening corrections) for the in-phase mode



This suggests: The most unstable mode lies at the edge of the 1st Brillouin zone

The above results imply that the bcc structure is **unstable** with respect to compressional perturbations at finite wavenumbers

Then, what is the most unstable direction of the wave vector?

Anisotropy of matter of the inner crust

- Crystal has cubic symmetry, and the elastic properties are anisotropic.
- Deformation vector field $\boldsymbol{u}(\boldsymbol{r})$.
- Deformation tensor field $u_{ij} = \partial u_i(\boldsymbol{r})/\partial r_j$.
- Energy of deformation of a cubic crystal to the 2nd order:

$$\delta F = \frac{1}{2}\lambda_{1111}\left(u_{11}^2 + u_{22}^2 + u_{33}^2\right) + \lambda_{1122}\left(u_{11}u_{22} + u_{11}u_{33} + u_{22}u_{33}\right) + 2\lambda_{1212}\left(u_{12}^2 + u_{13}^2 + u_{23}^2\right)$$

Stability of crystal

- General stability condition: positive definite dynamic matrix

$$\det\left(\sum_{b,c=1,2,3} \lambda_{abcd} k_b k_c\right) > 0$$

- Minimizing the *Gibbs free energy* of a crystal under external pressure is convenient because λ's retain the Voigt symmetry:

$$\lambda_{1212} = \lambda_{1221}$$

- Simplify notation:

$$\lambda_{1111} = C_{11}$$

$$\lambda_{1122} = C_{12}$$

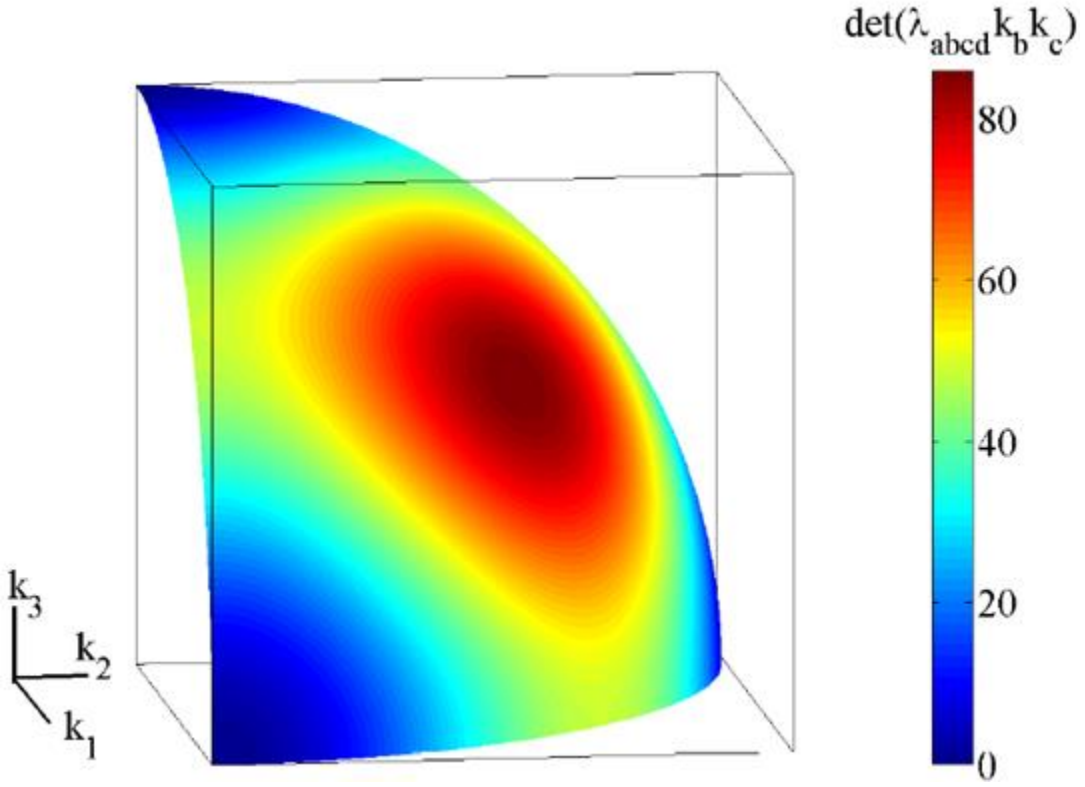
$$\lambda_{1212} = \lambda_{1221} = C_{44}$$

- Characteristics of Coulomb crystal have been calculated by Fuchs in 1936 [K. Fuchs, *Proc. Roy. Soc. London* **153**, 622 (1936)]:

$$\frac{c_{11} - c_{12}}{c_{44}} \approx 0.13$$

Then we find numerically:

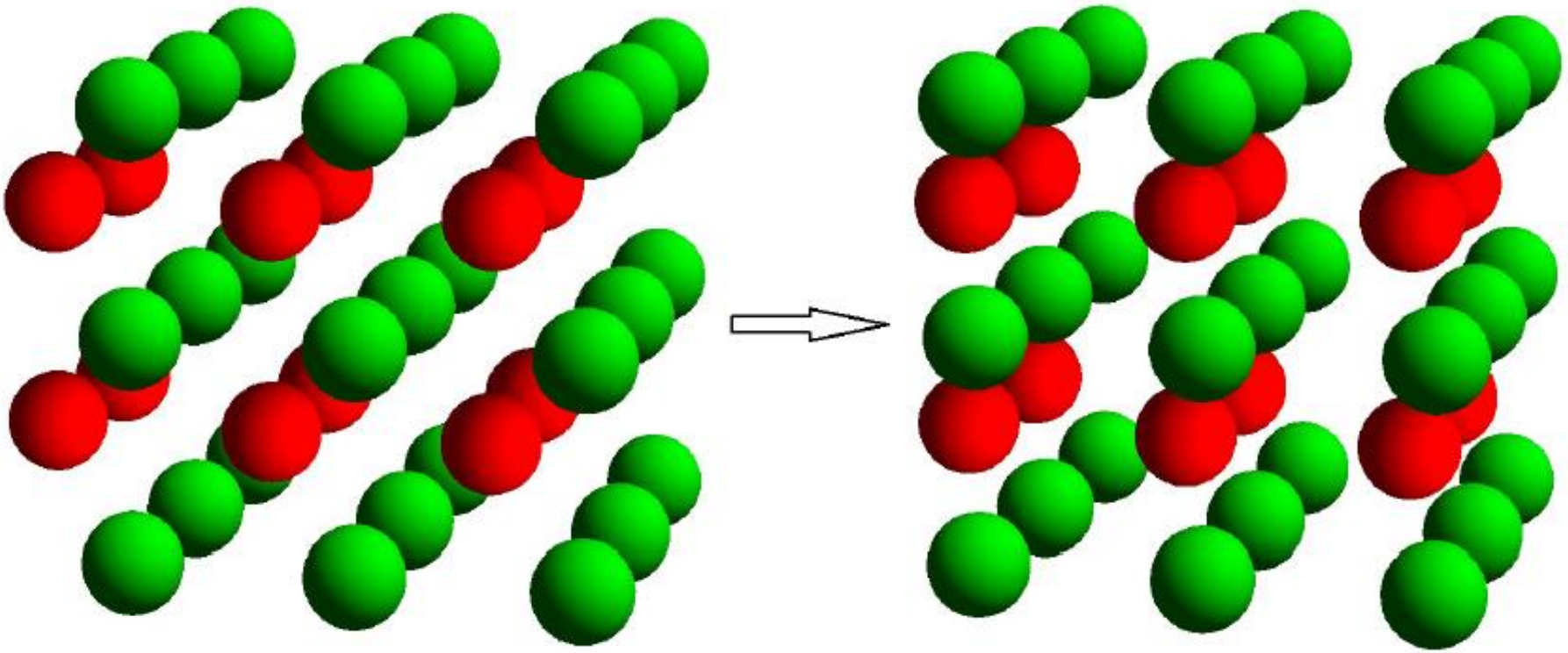
The most unstable direction



- This result was obtained by Cahn analytically for $C_{11} - C_{12} > 2C_{44}$ [J. Cahn, *Acta Metallurgica* **10**, 179 (1962)]

Possible final configuration

- Follows from the dispersion relation



Why is the inner crust important?

- Thermodynamic and transport properties (thermal and neutron conductivity, specific heat, …).
  Observationally:

  Cooling.

  Glitches.
- Elastic properties (elastic moduli, breaking strain, vibrational and torsional modes, …).
  Observationally:

  Precursors of γ-ray bursts in NS mergers.
  Generation of gravitational waves.

Robustness of our result

- Independence on the model of microscopic nuclear interactions [checked so far using two nuclear models: Lattimer-Swesty (our work), and MF-ETF with shell corrections (N. Chamel, private comm.)].
- Can be seen straightforwardly close to neutron drip:

$$E_{pp}(k) - \frac{E_{np}^2}{E_{nn}} = E_{pp}(k) - \frac{E_{np}^2}{E_{nn}^2} E_{nn}$$

$$\frac{E_{np}^2}{E_{nn}^2} \sim O(1)$$

$$E_{nn} \propto (n_n^{out})^{-1/3}$$

Conclusions

- We predict a structural transition in the crystal lattice of the inner crust: **the bcc structure is unstable**.
- Transition occurs on finite wavenumbers (*similarly to martensitic transition*), along a cubic axis (*similarly to spinodal decomposition*).
- To establish the final structure and calculate its properties we need nuclear calculations above the neutron drip, taking into account: neutron liquid, electron screening, small perturbations of the ion positions.

Additional slides

(1 of 4) *Why bcc is the equilibrium structure?

- The total energy

$$E = E_{Coul} + E_{curv} + \sum_{i=n,p,e,\mu} \mu_i$$

- Its perturbation

$$\delta E = \frac{1}{2!} V \sum_{\boldsymbol{Q}\boldsymbol{Q}'} v(Q) \delta_{\boldsymbol{Q}+\boldsymbol{Q}',0} \left(\delta n_p\right)^2 + \frac{1}{3!} V \sum_{\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{Q}''} v_3(\boldsymbol{Q},\boldsymbol{Q}',\boldsymbol{Q}'') \delta_{\boldsymbol{Q}+\boldsymbol{Q}'+\boldsymbol{Q}'',0} \left(\delta n_p\right)^3 + \ \ldots$$

***** Detailed discussion in
Nucl. Phys. A **175**, 225 (1971)
G. Baym, H. A. Bethe, C. J. Pethick

(2 of 4) *Why bcc is the equilibrium structure?

- Count the reciprocal vectors satisfying $\boldsymbol{Q} + \boldsymbol{Q}' = 0$
- Start with bcc structure

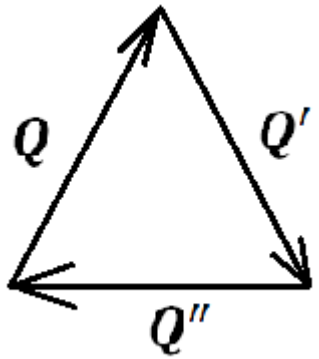
$$\left.\begin{array}{l} \frac{1}{2}(0 \quad \pm 1 \quad \{\pm 1 \; or \mp 1\}), \quad \frac{1}{2}(\pm 1 \quad 0 \quad \{\pm 1 \; or \mp 1\}), \\ \qquad\qquad\qquad\qquad \frac{1}{2}(\pm 1 \quad \{\pm 1 \; or \mp 1\} \quad 0) \end{array}\right\} 12$$

Thus

$$\frac{1}{2!} V \sum_{\boldsymbol{Q}\boldsymbol{Q}'} v(Q) \delta_{\boldsymbol{Q}+\boldsymbol{Q}',0} \left(\delta n_p\right)^2 = 6v(Q)\left(\delta n_p\right)^2$$

(3 of 4) *Why bcc is the equilibrium structure?

- Count the reciprocal vectors satisfying $\boldsymbol{Q} + \boldsymbol{Q}' + \boldsymbol{Q}'' = 0$; these are: $\frac{1}{2}(1 \quad 1 \quad 0)$, $\frac{1}{2}(-1 \quad 0 \quad 1)$, $\frac{1}{2}(0 \quad -1 \quad -1)$ and $\pm$. For each of 12 choices of $\boldsymbol{Q}$, there are 4 choices for $\boldsymbol{Q}'$, and 1 choice for $\boldsymbol{Q}''$. Total number is 48.



Thus: $$\frac{1}{3!}V\sum_{\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{Q}''}\delta(\boldsymbol{Q}+\boldsymbol{Q}'+\boldsymbol{Q}'')v^{(3)}(\boldsymbol{Q},\boldsymbol{Q}',\boldsymbol{Q}'')(\delta n_p)^3 = 8v^{(3)}(Q)(\delta n_p)^3$$

(4 of 4) *Why bcc is the equilibrium structure?

- Thus, for bcc: $\delta E = \left(6v(Q) + 8v^{(3)}(Q)\delta n_p\right)\left(\delta n_p\right)^2$.
- For fcc and sc: $\delta^{(3)}E = 0$.
- For hcp: $\delta E = \frac{2}{3}\left(6v(Q) + 3v^{(3)}(Q)\delta n_p\right)\left(\delta n_p\right)^2$.
- Conclusion: for infinetisemal δn_p, the bcc phase is the most energetically favorable static configuration of Coulomb crystals.