Dual variables at work: the 1+1-dimensional O(3) model

Falk Bruckmann
(Regensburg University)

Trento, Oct. 2015

with Tin Sulejmanpasic [1408.2229]
and Thomas Kloiber, Christof Gattringer [1507.04253, in prep.]

Heisenberg-Programm
Deutsche Forschungsgemeinschaft

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Physics motivation

the $O(3)$ model in 1+1 dimensions

$$S = \frac{1}{g^2} \int d^2x \, \frac{1}{2} (\partial_\nu n_a)^2 \,, \quad n_a^2 = 1 \,, \quad a = 1,2,3 \,, \quad \nu = 1,2$$

rich variety of effects, all similar to QuantumChromoDynamics:

- no dimensionful parameter, but dynamical mass generation
- asymptotic freedom: strongly coupled in the infrared
- topology & instantons
- lowest model of the $O(N)$ and $CP(N)$ series, large-N possible
  $O(3)$ best compared to $SU(2)$ Yang-Mills
- fermions can be coupled
- lattice formulation = Heisenberg model: $J \sum_{x,\nu} n_a(x) n_a(x+\hat{\nu})$
- global symmetry $\Rightarrow$ conserved charges

Thermodynamics

Euclidean path integrals

nonzero temperature $\Rightarrow$ base space $R^1 \cdot S^1_{1/T}$ (corresponding lattice)

- here: low temperatures = large temporal extent

nonzero density/chemical potential $\Rightarrow$ action modified:

- one of the O(3) generators rotates $n_{1,2}$

  conserved current

  couple μ to that symmetry (other μ couplings equivalent)

  new action contains μQ (= charge), but in bosonic theories also μ^2

- continuum: $g^2\mathcal{L} = \frac{1}{2}(\partial_\nu \vec{n})^2 + \underbrace{i\mu(n_1 \dot{n}_2 - n_2 \dot{n}_1)}_{\text{complex}} \underbrace{- \frac{\mu^2}{2}(n_1^2 + n_2^2)}_{+\frac{\mu^2}{2}(n_3^2 - 1)}$

$\Rightarrow$ suppression of the perpendicular component, effectively $O(2)$ like with magnetic field

$\Rightarrow$ complex action/sign problem: importance sampling?? note that imaginary μ is fine

Physics expectations

(i) charged particles induced only after μ has reached the mass gap

- = partition function remains μ-indep. up to $\mu_c = m$[1]

  'Silver-Blaze' region & transition

- measuring μ_c (having solved the sign problem) one gets information about the mass of the charged particles

  for other interesting insights see the talk by T. Sulejmanpasic

(ii) for large μ the system becomes more and more O(2)-like

- vortices
- condensing or in vortex-antivortex pairs depending on the coupling = Berezinskii-Kosterlitz-Thouless transition

  no order parameter singular, two point correlators change

[1] QCD: $\mu_{\text{up}} = \mu_{\text{down}} \Rightarrow 3\mu_c = m_{\text{lightest baryon}}$, $\mu_{\text{up}} = -\mu_{\text{down}} \Rightarrow 2\mu_c = m_{\text{pion}}$

Dualization in a nutshell

- lattice action with $n_1 + in_2 = n_{12} e^{i\phi}$: $\quad (n_{12}^2 + n_3^2 = 1$, spacing = 1)

$$S = -J \sum_{x,\nu} \Big(n_3(x) n_3(x+\hat{\nu}) + \frac{1}{2} n_{12}(x) n_{12}(x+\hat{\nu}) \cdot \\ \cdot \left\{ e^{i(\phi(x+\hat{\nu})-\phi(x))} e^{\mu\,\delta_{\nu,0}} + \underbrace{e^{-i(\phi(x+\hat{\nu})-\phi(x))} e^{-\mu\,\delta_{\nu,0}}}_{\text{not c.c.}} \right\} \Big)$$

- idea: for all terms A in the action, expand $e^{-S_A[n]}$ with variables k_A
  then integrate out the original fields n
- result: the system is now represented in 'dual variables' k
  typically constrained, simulated by worm algorithms
  provided all weights are positive = sign problem solved

Dualization in a nutshell

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$$S = -J \sum_{x,\nu} \Big(n_3(x) n_3(x+\hat\nu) + \frac{1}{2} n_{12}(x) n_{12}(x+\hat\nu) \cdot \\ \cdot \left\{ e^{i(\phi(x+\hat\nu)-\phi(x))} e^{\mu\,\delta_{\nu,0}} + e^{-i(\phi(x+\hat\nu)-\phi(x))} e^{-\mu\,\delta_{\nu,0}} \right\} \Big)$$

- bond variables for the three terms above

  e.g. for the third component:

$$e^{-S} \propto \prod_{x,\nu} e^{J\, n_3(x) n_3(x+\hat\nu)} = \prod_{x,\nu} \sum_{k_\nu(x)=0}^{\infty} \frac{J^{k_\nu(x)}}{k_\nu(x)!} n_3(x)^{k_\nu(x)} n_3(x+\hat\nu)^{k_\nu(x)}$$

- J comes with a power of all dual variables

Dualization in a nutshell

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$$S = -J \sum_{x,\nu} \Big(n_3(x) n_3(x+\hat{\nu}) + \frac{1}{2} n_{12}(x) n_{12}(x+\hat{\nu}) \cdot$$
$$\cdot \left\{ e^{i(\phi(x+\hat{\nu})-\phi(x))} e^{\mu\,\delta_{\nu,0}} + e^{-i(\phi(x+\hat{\nu})-\phi(x))} e^{-\mu\,\delta_{\nu,0}} \right\} \Big)$$

- integrations over $\{n_3(x), n_{12}(x), \phi(x)\} = \{\cos\theta(x), \sin\theta(x), \phi(x)\}$
  the latter gives constraints:

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi(x)\, e^{i\phi(x) \sum_\nu [m_\nu(x+\hat{\nu}) - m_\nu(x)]} \equiv \delta_{\text{Kron}}(\nabla_\nu m_\nu) \quad \forall x$$

  namely that $m_\nu \in \mathbb{Z}$ is divergence-free
  (m_ν is the difference of two nonnegative dual variables)

$m_\nu \in \mathbb{Z}$ is divergence-free

- symmetry manifest
- obviously the conserved charge reads $Q = \sum_{x_1} m_0(x_0, x_1)$

$\Rightarrow$ on each time slice the net flux of the dual variable m_ν is the charge Q

- the coupling of the chemical potential is done via

$$e^{-\mu \sum_{x_1} m_0(x_0, x_1)} > 0$$

if the total weight of the dual variables m_ν etc. is positive at $\mu = 0$ ✓ it remains so at $\mu \neq 0$

$\Rightarrow$ sign problem solved

note that imaginary μ is bad

Full result and interpretation, in O(4)

$$Z = \sum_{\substack{m\in\mathbb{Z} \\ \bar{m},k^{(3)},k^{(4)}\in\mathbb{N}_0}} \prod_{x,\nu} \frac{(J/2)^{|m_\nu(x)|+2\bar{m}_\nu(x)} J^{k_\nu^{(3)}(x)+k_\nu^{(4)}(x)}}{(|m_\nu^{(j)}(x)|+\bar{m}_\nu(x))!\,\bar{m}_\nu(x)!k_\nu^{(3)}(x)!k_\nu^{(4)}(x)!} \qquad \text{(expans.)}$$

$$\times \prod_x \frac{\Gamma(1+\frac{a(x)}{2})\Gamma(\frac{1}{2}+\frac{b^{(3)}(x)}{2})\Gamma(\frac{1}{2}+\frac{b^{(4)}(x)}{2})}{\Gamma(1+\frac{a(x)}{2}+\frac{1}{2}+\frac{b^{(3)}(x)}{2}+\frac{1}{2}+\frac{b^{(4)}(x)}{2})} \qquad \text{(integrated fields)}$$

$$\times \prod_x \delta\big(\nabla_\nu m_\nu \cdot e^{-\mu\sum_{x_1} m_0}\big) \qquad \text{(constraints and chem. potential)}$$

$$\times \prod_x \delta_{\text{even}}\big(b^{(3)}(x)\big)\delta_{\text{even}}\big(b^{(4)}(x)\big) \qquad \text{(evenness constraints)}$$

---

where

$$a(x) = \sum_\nu \big[|m_\nu(x)| + |m_\nu(x+\hat{\nu})| + 2\bar{m}_\nu(x) + 2\bar{m}_\nu(x+\hat{\nu})\big]$$

$$b^{(j)}(x) = \sum_\nu \big[k_\nu^{(j)}(x) + k_\nu^{(j)}(x+\hat{\nu})\big] \qquad j = 3,\ldots,N$$

Full result and interpretation, in O(4)

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$$\times \prod_x \delta_{\text{even}}\left(b^{(3)}(x)\right)\delta_{\text{even}}\left(b^{(4)}(x)\right) \qquad \text{(evenness constraints)}$$

- second line = ‘beta function’
- fix length of original field by Lagrange multiplier:

  numerator Γ’s from integrating out unconstrained fields, denominator Γ from integrating out Lagrange multiplier
- mass gap derived in original theory at large-N with Lagrange mult.

Full result and interpretation, in O(4)

$$Z = \sum_{\substack{m\in\mathbb{Z} \\ \bar{m},k^{(3)},k^{(4)}\in\mathbb{N}_0}} \prod_{x,\nu} \frac{(J/2)^{|m_\nu(x)|+2\bar{m}_\nu(x)} J^{k_\nu^{(3)}(x)+k_\nu^{(4)}(x)}}{(|m_\nu^{(j)}(x)|+\bar{m}_\nu(x))!\,\bar{m}_\nu(x)!k_\nu^{(3)}(x)!k_\nu^{(4)}(x)!} \quad \text{(expans.)}$$

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$$\times \prod_x \delta\big(\nabla_\nu m_\nu \cdot e^{-\mu\sum_{x_1} m_0}\big) \quad \text{(constraints and chem. potential)}$$

$$\times \prod_x \delta_{\text{even}}\big(b^{(3)}(x)\big)\delta_{\text{even}}\big(b^{(4)}(x)\big) \quad \text{(evenness constraints)}$$

- third line: μ in (1,2)-components couples to conserved current manifest since parametrization with polar angle, adapted
- there must be a conserved current in e.g. the (3,4)-components not explicit in this representation (polar angle in (3,4)-component!)
- fourth line: more constraints (!)

Intermezzo: Full result for CP(N-1) models

- N chemical potentials $\mu^{(j)}$, $j = 1, \ldots, N$:

$$Z = \sum_{\substack{m_\nu^{(j)}(x) \in \mathbb{Z} \\ \bar{m}_\nu^{(j)}(x) \in \mathbb{N}_0}} \qquad \text{(2}N\text{ `colored' dual variables)}$$

$$\prod_{x,\nu,j} \frac{J^{|m_\nu^{(j)}(x)| + 2\bar{m}_\nu^{(j)}(x)}}{(|m_\nu^{(j)}(x)| + \bar{m}_\nu^{(j)}(x))!\, \bar{m}_\nu^{(j)}(x)!} \qquad \text{(exponent expansion)}$$

$$\times \prod_x \frac{\Gamma(1 + \frac{a^{(1)}(x)}{2})\Gamma(1 + \frac{a^{(2)}(x)}{2}) \ldots \Gamma(1 + \frac{a^{(N)}(x)}{2})}{\Gamma(N + \frac{a^{(1)}(x)}{2} + \frac{a^{(2)}(x)}{2} + \ldots + \frac{a^{(N)}(x)}{2})} \qquad \text{(integrated fields)}$$

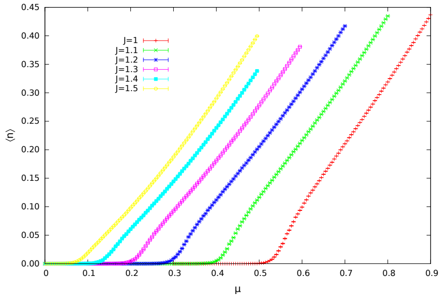
$$\times \prod_{x,j} \Big(\delta(\nabla_\nu m_\nu) \cdot e^{-\mu \sum_{x_1} m_0} \Big)^{(j)} \qquad \text{(constraints and chem. potentials)}$$

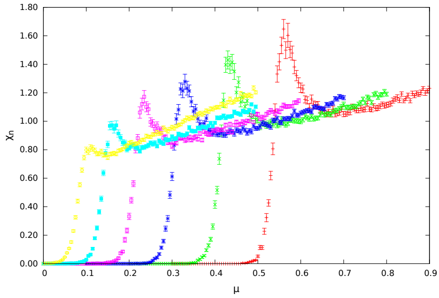
$$\times \prod_{x,\nu} \delta\Big(\sum_j m_\nu^{(j)}(x) \Big) \qquad \text{(local U(1) symmetry, on bonds)}$$

- no sign problem here either

Numerical results: Silver blaze transition

- particle number density and its susceptibility as functions of μ for different couplings J: (90×90 lattice)



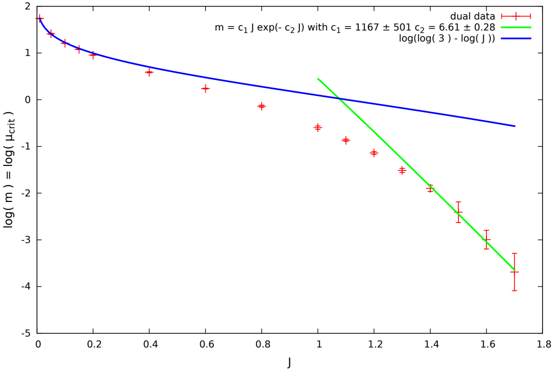


independent of μ up to some μ_c ✓

movie (10×1000)

- $\mu_c = m$ and expansions for mass (bare, in lattice units):

$$am = \begin{cases} -\log \frac{J}{3} & \text{strong coupling} \\ \#\exp(-2\pi J) & \text{weak coupling (large } N\text{)} \end{cases}$$

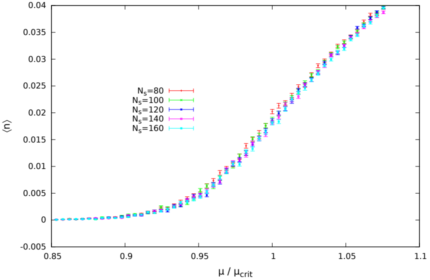


- also checked against conventional spectroscopy

Order of the transition

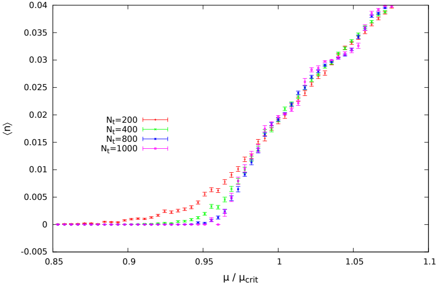
- at fixed J = fixed lattice spacing, scaling of $\langle Q \rangle$ with (J=1.3)

volume



$L = 18 \ldots 35 \frac{1}{m} \quad (T = 0.02m)$

temperature

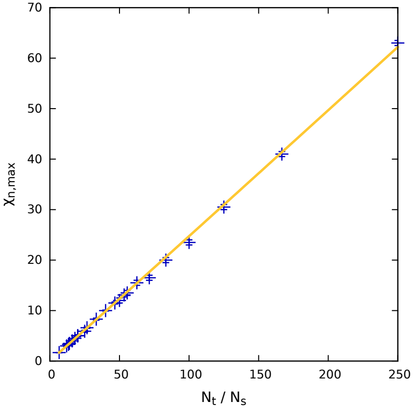


$T = 0.02 \ldots 0.005\, m \quad (L = 22\, \frac{1}{m})$

a crossover . . . . . . that sharpens as $T \to 0$

- at $T = 0$ non-analytic?! "quantum phase transition"

- scaling of the maximum of the particle number susceptibility χ with $1/LT = N_t/N_s$ (lattice aspect ratio):



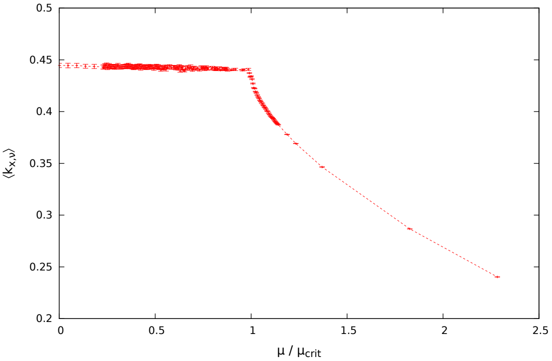
$$\chi_{\text{max}} \propto \frac{1}{(LT)^\gamma} \quad \text{with } \gamma = 0.998(3)$$

Nature of the transition

at large μ the system tends to be planar due to expl. breaking $+\frac{\mu^2}{2}n_3^2$

- out-of-plane hoppings become suppressed:

$$J\left\langle n_3(x)n_3(x+\hat{\nu})\right\rangle_{\text{orig}} = \left\langle k_\nu(x)\right\rangle_{\text{dual}}$$



(value at small μ $\rightleftharpoons$ hoppings isotropic ✓)

Two point correlator and vortices

in-plane:

$$C(x-y) = \langle e^{i\phi(x)} e^{-i\phi(y)} \rangle$$

- expectation in the O(2) model

$$\mathcal{L}_{O(2)} = -J \sum_{x,\nu} \cos(\phi(x+\hat{\nu}) - \phi(x))$$

th correlator decays exponentially/algebraically

$$C(x-y) \to \begin{cases} J^{|x-y|/a} & \text{for small } J = \text{strong coupling} \\ \frac{1}{|x-y|^{1/2\pi J}} & \text{for large } J = \text{weak coupling} \end{cases}$$

because vortices (also called merons)

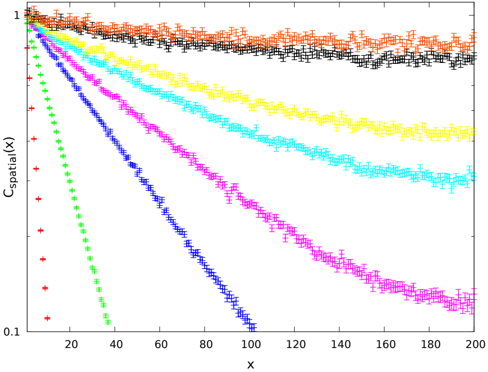
$$\begin{cases} \text{condense} & \text{for small } J \\ \text{come in vortex-antivortex pairs} & \text{for large } J \end{cases}$$

both $C(x-y) \overset{|x-y|\to\infty}{\longrightarrow} 0$: no ordered phase in 2d   Mermin-Wagner

- where will we land at $\mu > \mu_c$? $J_{O(2)} = J^{\text{eff}}_{O(3)}(\text{scale } \mu) \propto \log(\mu)$

  will we see BKT as a second transition?
- time-integrated spatial correlator ('zero-momentum') for the sake of high statistics

  logarithmically:



exp. decay lost above μ_c!? time integration spoils interpretation

Summary and outlook

- dualization of O(N) and CP(N-1) models with chemical potential(s)
  sign problem gone, actually in all dimensions
  (some) symmetries manifest
- more to learn from dual variables
  realization of mass gap?
  interplay of pert. expansion & nonpert. physics = resurgence?
  Dunne, Ünsal 12
- quantum phase transition? $\chi_{\mathrm{max}} \sim 1/(LT)^{0.998(3)}$
- Berezinskii-Kosterlitz-Thouless as a second transition?
  more data, on two point and vortex correlator
- outlook: topological angle θ, higher models & large-N