

Color superconductivity from the chiral quark-meson model

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Abstract

We study the two-flavor color superconductivity of low-temperature quark matter in the vicinity of chiral phase transition in the quark-meson model where the interactions between quarks are generated by pion and sigma exchanges. Starting from the Nambu-Gor'kov propagator in real-time formulation we obtain finite temperature (real axis) Eliashberg-type equations for the quark self-energies (gap functions) in terms of the in-medium spectral function of mesons. Exact numerical solutions of the coupled nonlinear integral equations for the real and imaginary parts of the gap function are obtained in the zero temperature limit using a model input spectral function. We find that these components of the gap display a complicated structure with the real part being strongly suppressed above $2\Delta_0$, where Δ_0 is its on-shell value. We find $\Delta_0 \simeq 40$ MeV close to the chiral phase transition.

Keywords: models of QCD, phase diagram of dense matter, color superconductivity

1. Introduction

Low-temperature quark matter at large chemical potential is expected to be a color superconductor [1, 2]. In its ground state, it forms a coherent state of bound Cooper pairs which flow without resistivity. At moderate densities, the most robust pairing pattern involves two light flavors of up and down quarks forming Cooper pairs with a wave-function that is antisymmetric in color space [3].

Experimental programs exploring highly compressed matter in heavy-ion collisions will probe the region of the phase diagram of strong interaction matter where the interplay between the chiral symmetry breaking and color superconductivity is an important factor [4]. In this regime of interest, which is close to the chiral phase transition line, quarks and mesons are the dominant degrees of freedom. Having this context in mind, we address here the 2SC pairing in quark matter in the quark-meson model, which is a renormalizable model that shares the chiral symmetry breaking pattern with the underlying fundamental theory of QCD [5, 6]. More specifically, our work is

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further motivated by the recent observation that the entropy of this model shows anomalies at low-temperatures, when studied within the functional renormalization group formalism [7]. This could be an indication of the instability of the obtained ground state toward color superconductivity.

Color superconductivity in the 2SC phase was studied at asymptotically high densities within perturbative QCD framework in Refs. [8–11]. In these theories, the interaction between quarks is mediated via (screened) gluon exchanges and the pairing fields are governed by Eliashberg-type equations, familiar from boson-exchange models of superconductivity. Approximate solutions of these equations for the case of massless quarks were obtained which exhibit the scaling of the gap (more precisely, its on-shell value Δ_0) with the strong coupling λ as $\Delta_0 \propto \exp(-1/\lambda)$; these solutions also identified the pre-factor of the (approximate) gap equation for the real part of the pairing field. However, to our knowledge, the effects of retardation of interaction via gluon exchange and the resulting complex nature of the gap function have not been exposed so far.

The aim of this work is thus to address again the problem of 2SC pairing, however within a model which is better suited in the regime close to the chiral phase transition and to maintain the complex nature of the gap throughout the calculation. We choose to work with the quark-meson model, where the interaction between quarks is mediated by pseudo-scalar pion exchanges and scalar sigma exchanges. The quarks are assumed to be massive due to the dynamical mechanism of chiral symmetry breaking. We find the equations for the 2SC pairing gap appropriate for the quark-meson model, which naturally encapsulate the information on the spectral functions of mesons. Furthermore, using an approximate form of the input spectral functions of mesons we solve the obtained Eliashberg-type equations exactly, thus fully exhibiting the complex nature of the pairing gap.

This paper is organized as follows. In Sec. 2 we set up the formalism for 2SC pairing with the quark-meson model and obtain the relevant equations for the pairing gap. Section 3 describes the results of numerical solutions of the gap equations. Our results are summarized in Sec. 4.

2. Formalism

In this work we apply the Nambu-Gorkov formalism where the quark states are combined in spinors (our notations follow Ref. [12])

$$\Psi \equiv \begin{pmatrix} \psi \\ \bar{\psi}^T \end{pmatrix} \equiv \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}.$$

The inverse quark propagator, defined in a standard fashion via the Nambu-Gorkov spinors Ψ , is given by

$$S^{-1}(q) = \begin{pmatrix} \not{q} + \mu\gamma_0 - m & \bar{\Delta} \\ \Delta & (\not{q} - \mu\gamma_0 + m)^T \end{pmatrix}, \quad (1)$$

where the following relation holds $\bar{\Delta} = \gamma_0 \Delta^\dagger \gamma_0$. We consider the case of equal number densities of up and down quarks with a common chemical potential μ and mass m . The real time-structure of the propagators and self-energies are not specified for simplicity until later. Furthermore, the vertex corrections to the quark-meson vertices $\Gamma_\pi^i(q)$ and $\Gamma_\sigma(q)$ will be neglected and these will be

approximated by their bare values

$$\Gamma_\pi^i(q) = \begin{pmatrix} \frac{\tau^i}{2}\gamma_5 & 0 \\ 0 & -(\frac{\tau^i}{2}\gamma_5)^T \end{pmatrix}, \quad \Gamma_\sigma(q) = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad (2)$$

where pions are assumed to couple via pseudo-scalar coupling and \mathbb{I} is a unit matrix in the Dirac and isospin spaces. The pion and sigma propagators are given by

$$D_\pi(q) = \frac{1}{q_0^2 - \mathbf{q}^2 - m_\pi^2}, \quad D_\sigma(q) = \frac{1}{q_0^2 - \mathbf{q}^2 - m_\sigma^2}, \quad (3)$$

where $m_{\pi/\sigma}$ are their masses. The gap equation for Δ in the Fock approximation is then given by

$$\begin{aligned} \Delta(k) &= ig_\pi^2 \int \frac{d^4q}{(2\pi)^4} \left(-\frac{\tau^i}{2}\gamma_5\right)^T S_{21}(q) \frac{\tau^j}{2}\gamma_5 \delta_{ij} D_\pi \\ &+ ig_\sigma^2 \int \frac{d^4q}{(2\pi)^4} (-\mathbb{I})^T S_{21}(q) \mathbb{I} D_\sigma(q-k), \end{aligned} \quad (4)$$

where g_π and g_σ are the coupling constants. The Ansatz for the gap in a 2SC superconductor is given by [3]

$$\Delta_{ij}^{ab}(k) = (\lambda_2)^{ab} (\tau_2)_{ij} C \gamma_5 [\Delta_+(k) \Lambda^+(k) + \Delta_-(k) \Lambda^-(k)], \quad (5)$$

where $a, b \dots$ refer to the color space, i, j, \dots refer to the flavor space and the projectors onto the positive and negative states are defined as $\Lambda^\pm(k) = (E_k^\pm + \boldsymbol{\alpha} \cdot \mathbf{k} + m\gamma_0)/2E_k^\pm$, where $E_k^\pm = \pm \sqrt{\mathbf{k}^2 + m^2}$ and $\boldsymbol{\alpha} = \gamma_0 \boldsymbol{\gamma}$. Inverting Eq. (1) one finds for the off-diagonal 21 component of the quark propagator

$$S_{21}(q) = -(\lambda_2 \tau_2 C \gamma_5) \left[\frac{\Delta_+ \Lambda_-(q)}{q_0^2 - (\epsilon_q - \mu)^2 - \Delta_+^2} + \frac{\Delta_- \Lambda_+(q)}{q_0^2 - (\epsilon_q + \mu)^2 - \Delta_-^2} \right] = -(\lambda_2 \tau_2 C \gamma_5) F_{21}(q). \quad (6)$$

On substituting Eqs. (5) and (6) into Eq. (4) and cancelling common terms we find

$$\begin{aligned} \Delta_1(k) \Lambda^+(k) + \Delta_2(k) \Lambda^-(k) &= -ig_\pi^2 \frac{3}{4} \int \frac{d^4q}{(2\pi)^4} \gamma_5 F_{21}(q) \gamma_5 D_\pi(q-k) \\ &+ ig_\sigma^2 \int \frac{d^4q}{(2\pi)^4} F_{21}(q) D_\sigma(q-k). \end{aligned} \quad (7)$$

In the next step we decompose the remainder of the anomalous propagator into a sum of positive and negative state contributions $F_{21} = \Lambda^- f_1 + \Lambda^+ f_2$. Now, on multiply (7) from the right by $\Lambda^+(k)$ and $\Lambda^-(k)$, using the properties $(\Lambda^\pm)^2 = \Lambda^\pm$, $\Lambda^+ + \Lambda^- = 1$, $\Lambda^+ \Lambda^- = 0$, and taking the trace of the

resulting two equations (note that $\text{Tr } \Lambda^\pm = 4$) we obtain two gap equations introduced in Eq. (5)

$$\begin{aligned}\Delta_+(k) &= -\frac{3ig_\pi^2}{4} \int \frac{d^4q}{(2\pi)^4} (K_{-+}f_1 + K_{++}f_2)D_\pi(q-k) \\ &+ \frac{ig_\sigma^2}{4} \int \frac{d^4q}{(2\pi)^4} (M_{-+}f_1 + M_{++}f_2)D_\sigma(q-k).\end{aligned}\quad (8)$$

$$\begin{aligned}\Delta_-(k) &= -i\frac{3g_\pi^2}{4} \int \frac{d^4q}{(2\pi)^4} (K_{--}f_1 + K_{+-}f_2)D_\pi(q-k) \\ &+ i\frac{g_\sigma^2}{4} \int \frac{d^4q}{(2\pi)^4} (M_{--}f_1 + M_{+-}f_2)D_\sigma(q-k)\end{aligned}\quad (9)$$

where $K_{\pm\pm} = \text{Tr}[\gamma_5\Lambda^\pm(q)\gamma_5\Lambda^\pm(k)]$ and $M_{\pm\pm} = \text{Tr}[\Lambda^\pm(q)\Lambda^\pm(k)]$. The commutation property $[\Lambda^\pm, \gamma^5] = 0$ implies that we may set in Eqs. (8) and (9) $K_{\pm\pm} = M_{\pm\pm}$. A further simplification arises because one is generally interested in the gap at the Fermi surface of the particles and it is legitimate to drop the antiparticle component of the decomposition of the gap function (5) and take $\Delta_- = 0$. We find then

$$\Delta_+(k) = -i\frac{3g_\pi^2}{4} \int \frac{d^4q}{(2\pi)^4} K_{-+}f_1(k-q)D_\pi(q) + i\frac{g_\sigma^2}{4} \int \frac{d^4q}{(2\pi)^4} K_{-+}f_1(k-q)D_\sigma(q),\quad (10)$$

where $f_1 = \Delta_+/(q_0^2 - \xi_q^2 - \Delta_+^2)$ with $\xi_q = \varepsilon_q - \mu$. At this point we make explicit the finite-temperature content of the equations above within the Schwinger-Keldysh real-time formalism. The retarded component of the gap function can be written in standard notations [13, 14]

$$\Delta_+^R(k_0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{D^>(\omega')F^>(\omega-\omega') - D^<(\omega')F^<(\omega-\omega')}{k_0 - \omega + i\eta},\quad (11)$$

where

$$F^{>,<}(p) = iA(p)f^{>,<}(p), \quad f^<(p) = n_F(p), \quad f^>(p) = 1 - n_F(p),\quad (12)$$

$$D^{>,<}(q) = iB(q)g^{>,<}(q), \quad g^<(q) = n_B(q), \quad g^>(q) = 1 + n_B(q),\quad (13)$$

$n_{F/B}(p)$ are the Fermi and Bose distribution functions, $A(p)$ and $B(p)$ are the fermionic and bosonic spectral functions; we we have suppressed in these equations the pion and sigma indices and momentum variables which will be restored below. In terms of these functions Eq. (10) can be written as

$$\begin{aligned}\Delta_+^R(k_0, \mathbf{k}) &= -i\frac{3g_\pi^2}{4} \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} d\varepsilon A(\varepsilon, \mathbf{k}-\mathbf{q}) \int_0^{\infty} \frac{d\omega'}{2\pi} B_\pi(\omega') J_\pi(k_0, \omega', \varepsilon) K_{-+} \\ &+ i\frac{g_\sigma^2}{4} \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\infty} d\varepsilon A(\varepsilon, \mathbf{k}-\mathbf{q}) \int_0^{\infty} \frac{d\omega'}{2\pi} B_\sigma(\omega') J_\sigma(k_0, \omega', \varepsilon) K_{-+},\end{aligned}\quad (14)$$

where

$$J_{\pi/\sigma}(k_0, \omega, \varepsilon) = \frac{n_{B\pi/\sigma}(\omega) + n_F(\varepsilon)}{\varepsilon - k_0 - \omega - i\eta} + \frac{1 + n_{B\pi/\sigma}(\omega) - n_F(\varepsilon)}{\varepsilon - k_0 + \omega - i\eta} \simeq \frac{\theta(-\varepsilon)}{\varepsilon - k_0 - \omega - i\eta} + \frac{\theta(+\varepsilon)}{\varepsilon - k_0 + \omega - i\eta},\quad (15)$$

and the second approximate relation follows in the zero temperature limit to be used below. (From now on we drop the sub- and superscripts on Δ as we refer only to its retarded, positive energy component). In the zero temperature limit, the d^3q phase space integration can be transformed into an integration over the magnitude of q and the on-shell energy ξ_p , which can be then performed analytically. As a result we find

$$\Delta(k_0, k_F) = \int_0^\infty d\varepsilon \mathcal{F}(\varepsilon) \int_0^\infty d\omega' \lambda(\omega') \left[\frac{1}{\varepsilon + k_0 + \omega' + i\eta} + \frac{1}{\varepsilon - k_0 + \omega' - i\eta} \right], \quad (16)$$

where the kernel of the gap equation is given by

$$\lambda(\omega) = \frac{g_\sigma^2 \mathcal{B}_\sigma(\omega) - 3g_\pi^2 \mathcal{B}_\pi(\omega)}{4v_F}, \quad (17)$$

where v_F is the Fermi velocity of quarks and

$$\mathcal{B}_{\pi/\sigma}(\omega) = \int_0^{2k_F} \frac{qdq}{(2\pi)^2} B_{\pi/\sigma}(\omega, q) K_{-+}, \quad \mathcal{F}(\varepsilon) = \text{Re} \frac{\Delta(\varepsilon) \text{sgn}(\varepsilon)}{[\varepsilon^2 - \Delta(\varepsilon)^2]^{1/2}}. \quad (18)$$

In the case where the spectral function $B_{\pi/\sigma}(\omega, q)$ does not depend on the momentum transfer q , the first equation in (18) simplifies to $\mathcal{B}_{\pi/\sigma}(\omega) \simeq (m^2 k_F^2 / \pi^2 E_F^2) B_{\pi/\sigma}(\omega)$, where we substituted the zero temperature limit of $K_{-+}(q \rightarrow 0)$. Then, the kernel can be written as

$$\lambda(\omega) = (m^2 v_F / 4\pi^2) \left[g_\sigma^2 B_\sigma(\omega) - 3g_\pi^2 B_\pi(\omega) \right], \quad (19)$$

i.e., up to a constant factor, it is given by the sum of the spectral functions of mesons.

3. Numerical results

Eq. (16) represents two coupled non-linear integral equations for the real and imaginary parts of the gap function, which were solved iteratively on a quadratic mesh spanned by the variables $[\varepsilon, \omega]$. The numerical method has been described elsewhere [14]. We approximate the kernel of the gap function, Eq. (19), by a suitable Gaussian function of the form

$$\lambda(\omega) = \frac{g\omega}{(\omega - \omega_0)^2 + \gamma^2/4}, \quad (20)$$

with the parameter values chosen as $\gamma = 0.0972$, $g = 0.0077$ and $\omega_0 = 0.1734$. To obtain these parameter values we have computed Eq. (19) using as an input the spectral functions $B_{\pi/\sigma}(\omega)$ derived from the quark-meson model [15]. The centroid of Eq. (20) ω_0 is at the mass of the σ meson and its height g was matched to the numerical computation of Eq. (19). To explore the sensitivity of the result on the strength of the coupling we repeated the computations by rescaling $g \rightarrow \eta g$, where η is a constant factor. In Fig. 1 we plot the function $\lambda(\omega)$ in Eq. (20) for two values of η indicated in the figure.

The solutions of the gap equation are shown in Fig. 2, where we display the real and imaginary parts of the gap as a function of frequency. The on-shell value of the gap Δ_0 follows in the limit

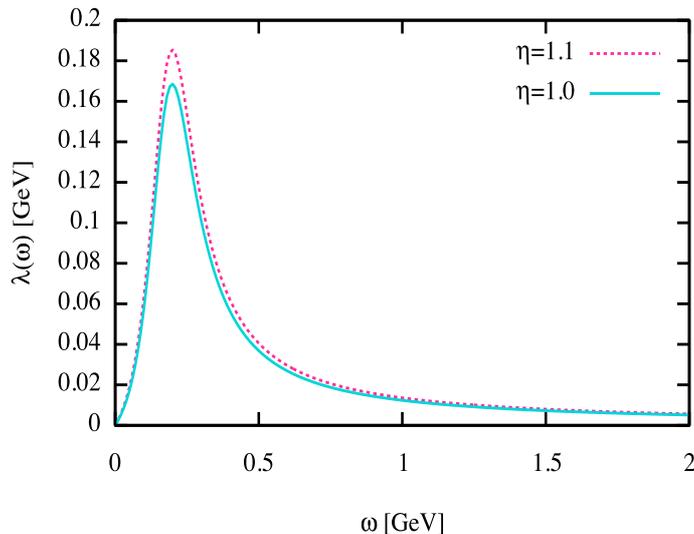


Figure 1: Dependence of the kernel function given by (20) for two values of the strengths of interaction parameterized by η on frequency.

$\omega = 0$ where it becomes purely real; it is seen that this value is rather sensitive to the strength parameter η . Increasing its value by 10% produces a four-fold increase in Δ_0 . Computations for a larger value $\eta = 1.4$ (not shown in the figure) display a further increase of the gap value up to $\Delta_0 \simeq 0.3$ GeV.

In the off-shell region, the imaginary and real parts of the gap show non-trivial structures. They intersect for $\omega \simeq 2\Delta_0$, beyond which the imaginary component dominates before both components vanish at asymptotically large frequencies. Note that in the ordinary BCS formulations the gap is real and constant in the off-shell region. Clearly, our results show that the constant gap approximation could be accurate only very close to the on-shell ($\omega \rightarrow 0$) limit. A proper account of the frequency dependence of the propagators of the color superconductors may be of importance for many frequency dependent observables, for example, for the description of their dynamical response to various perturbations. Examples include the dynamical (frequency dependent) Meissner effect or transport coefficients, such as shear viscosity. We recall that in the framework of the Kubo formalism, see e.g. [16], the last quantity requires an evaluation of the frequency derivative of response function, which will obtain an additional contribution through the frequency dependence of the gap function.

4. Conclusions and perspectives

We have set up a formalism to compute the pairing gap in the 2SC phase of low-temperature quark matter within the quark-meson model. Starting from the Nambu-Gorkov propagator of the quarks for the 2SC phase we have evaluated their anomalous self-energy (gap) due to meson exchanges. Using the real-time formalism we have expressed the gap function in terms of spectral functions of mesons (here pions and sigmas) at finite temperatures, see Eq. (14). The frequency

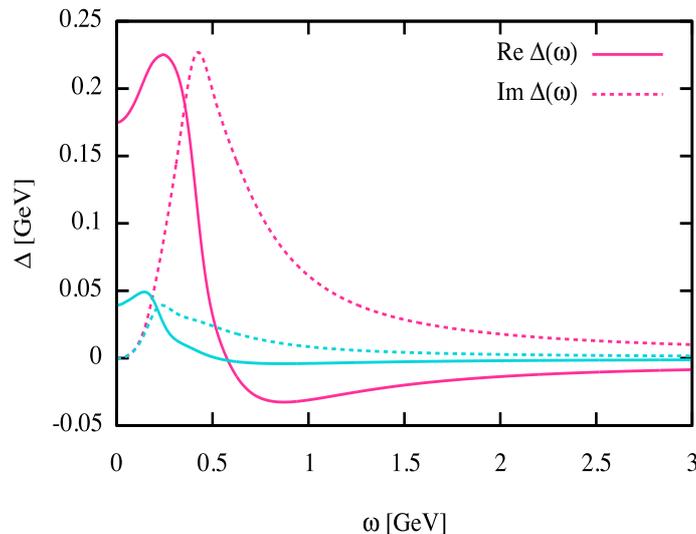


Figure 2: Dependence of the real (solid) and imaginary (dashed) components of the gap function on frequency for two value strengths of interaction $\eta = 1$ (lower) and $\eta = 1.1$ (upper) pair of curves.

dependence of the spectral functions implies a complex gap function, which physically reflects the retardation of the pairing interaction (which is absent in the BCS-type formulations). We have solved the coupled integral equations for the real and imaginary parts of the gap function in the zero-temperature limit, showing that these components have non-trivial structures in the frequency domain, see Fig. 2.

For the sake of physical insight and simplicity, we have approximated the full spectral functions of the quark-meson model by a Gaussian-type function and explored the dependence of the gap on the strength of the interaction. We find that the on-shell value of the gap strongly depends on the strengths of the attraction in the pairing channel, which is consistent with the expectations from the BCS type approaches. It would be interesting to evaluate the components of the 2SC gap function using spectral functions of the quark-meson model directly for specific values of density and temperature of quark matter.

The frequency dependence and complex nature of the gap function implies that a number of physical quantities may differ qualitatively from their BCS counterparts computed with a real, constant in the frequency domain, gap. Among many examples, the transport coefficients, such as shear viscosity [16–18], would be interesting to evaluate.

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